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Fuzzy regression methods – a comparative assessment

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Abstract

The fundamental differences between fuzzy regression and ordinary regression are identified in this paper. Fuzzy regression can be used to fit fuzzy data and crisp data into a regression model, whereas ordinary regression can only fit crisp data. Through a comprehensive literature review, three approaches of fuzzy regression are summarized. The first approach of fuzzy regression is based on minimizing fuzziness as an optimal criterion. The second approach uses least-squares of errors as a fitting criterion, and two methods are summarized in this paper. The third approach can be described as an interval regression analysis. For each fuzzy regression method, numerical examples and graphical presentations are used to evaluate their characteristic and differences with ordinary least-squares regression. Based on the comparative assessment, the fundamental differences between ordinary least-squares regression and conventional fuzzy regression are concluded – that is, ordinary least-squares regression modeling data with randomness type of uncertainty, and conventional fuzzy regression modeling data with fuzziness type of uncertainty. In order to integrate both randomness and fuzziness types of uncertainty into one regression model, the concept of hybrid fuzzy least-squares regression analysis is proposed in this paper, and the details of its method are derived in the accompanying paper.

Keywords: Fuzzy regression; Fuzziness; Hybrid fuzzy least-squares regression; Interval regression; Ordinary least-squares regression; Randomness; Uncertainty modeling

1. Introduction

Regression analysis is one of the most used statistical tools by engineers and scientists. Methods of regression analysis are commonly used to build a model using collected data containing uncertainties and to obtain a prediction equation for the entire population. The computations of regression analysis can be conveniently performed by computer programs. Since most digital computers can only process ordinary crisp numbers, if linguistic data are obtained, symbolic numbers are used to represent qualitative terms, for examples, number 4 for "excellent", 3 for "very good", 2 for "good", and 1 for "fair". In many real world problems, an oversimplification of data could leave out important information for regression models. Some observations can be described only in linguistic terms (such as fair, good, and excellent). For such data, fuzzy set theory provides a means for modeling such linguistic variables utilizing fuzzy membership functions. And, fuzzy regression was proposed to deal with fuzzy data. In contrast to the ordinary regression that is based on

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probability theory, fuzzy regression can be based on possibility theory [7] and fuzzy set theory [17].

In ordinary regression analysis, the unfitted errors between a regression model and observed data are generally assumed as observation error that is a random variable having a normal distribution, constant variance, and a zero mean. In fuzzy regression analysis, the same unfitted errors are viewed as the fuzziness of the model structure as was initially developed by Tanaka et al. [16]. Since then, other fuzzy regression methods have been developed using different optimal criteria for fuzzy line or curve fitting. Other contributions in this area are by Celminš [1,2], Diamond [6], Tanaka [13], Tanaka and Ishibuchi [14], Savic and Pedrycz [12] and Ishibuchi [8]. In this paper, three different approaches of fuzzy regression are summarized. First, a fuzzy regression method that is based on minimizing fuzziness for model-fitting is summarized. Second, fuzzy regression using least squares of errors as a criterion is introduced. Two methods under this category are summarized. Third, fuzzy regression is implemented by using interval analysis.

Numerical examples are presented for each fuzzy regression method. Since fuzzy regression can accept crisp data, fuzzy data or a possible mixture of crisp and fuzzy data, examples of crisp independent (or predictor) X and crisp dependent (or criterion) Y are presented, followed by examples of crisp X and fuzzy Y.

Through comparing the results of numerical examples, the objective of this paper is identifying the features and limitations of existing fuzzy regression methods. They are presented along with conclusions at the end of this paper.

2. Fuzzy regression using minimum fuzziness criterion

In fuzzy regression, deviations between observed values and estimated values are assumed to be due to system fuzziness or fuzziness of regression coefficients [16]. This assumption is shared by fuzzy regression methods described in this paper. The goal of fuzzy regression analysis is to find a regression model that fits all observed fuzzy data within a specified fitting criterion. Different fuzzy regression models are obtained depending on the fitting criterion used.

Tanaka et al. [16] proposed the first linear regression analysis with a fuzzy model. According to this method, the regression coefficients are fuzzy numbers, which can be expressed as interval numbers with membership values. Since the regression coefficients are fuzzy numbers, the estimated dependent variable $\hat{Y}$ is also a fuzzy number. A fuzzy regression analysis with only one independent variable X results in the following bivariate regression model:

$$\hat{Y} = \tilde{A}_0 + \tilde{A}_1 X, \tag{1}$$

where $\tilde{A}_0$ is a fuzzy intercept coefficient, and $\tilde{A}_i$ is a fuzzy slope coefficient. Each fuzzy parameter $\tilde{A}_i = (m_i, c_i)$ is expressed as symmetrical triangular membership function, which consists of fuzzy center m_i and fuzzy half-width c_i. Other membership-function forms can be used as well.

According to this approach, the fuzzy coefficients $\tilde{A}_i\,(i = 0, 1)$ are determined such that the estimated fuzzy output $\hat{Y}$ has the minimum fuzzy width, while satisfying a target degree of belief h. The term h is referred to as a measure of goodness of fit or a measure of compatibility between data and a regression model. Each of the observed data sets, which can be fuzzy $\tilde{Y}_i$ or crisp datum Y_i, must fall within the estimated $\hat{Y}$ at h level as shown in Fig. 1. To determine the fuzzy coefficients $\tilde{A}_i = (m_i, c_i)$, Tanaka et al. [16] formulated the fuzzy regression objective as the following linear programming problem:

$$\text{minimize} \quad S = nc_0 + c_1 \sum_{i=1}^{n} |X_i| \tag{2}$$

$$\text{subject to} \quad c_0 \geqslant 0, \; c_1 \geqslant 0,$$

$$\sum_{j=0}^{l} m_i X_{ij} + (1-h) \sum_{j=0}^{l} c_i |X_{ij}| \geqslant Y_i + (1-h)e_i \quad \text{for } i = 1 \text{ to } n, \tag{3}$$

$$\sum_{j=0}^{l} m_i X_{ij} - (1-h) \sum_{j=0}^{l} c_i |X_{ij}| \leqslant Y_i - (1-h)e_i \quad \text{for } i = 1 \text{ to } n, \tag{4}$$

where S is the total fuzziness of the regression model. Eqs. (3) and (4) can deal with observed fuzzy datum

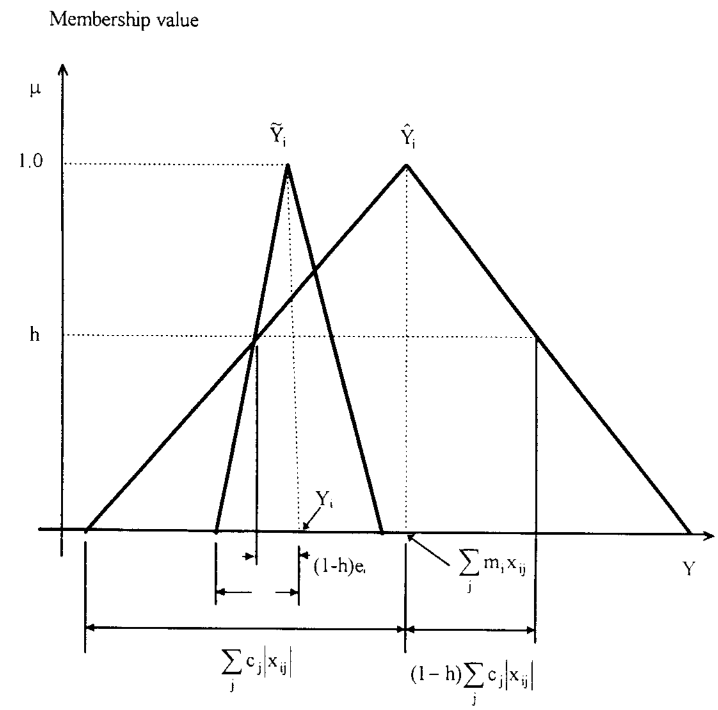

Fig. 1. Degree of fitting $\hat{Y}_i$ to given fuzzy datum $\tilde{Y}_i$.

$\tilde{Y}_i = (Y_i, e_i)$, where Y_i is the fuzzy center, and e_i is the fuzzy half-width. If an observed datum is crisp, its e is zero. Therefore, an ordinary crisp number is a special case of a fuzzy number.

Since the first fuzzy regression method was proposed, it drew criticism from Celminš [1], Chang et al. [5] and Redden and Woodall [11]. Following the introduction of the first method, several improved versions of fuzzy regression using the minimum fuzziness criterion were proposed. Tanaka and Ishibuchi [14] introduced quadratic membership functions to obtain fuzzy coefficients. Chang and Lee [4] proposed a method for fuzzy regression with widths unrestricted in sign. Tanaka et al. [15] proposed an exponential possibility regression, in which the result of fuzzy regression corresponds to the probabilistic regression. In these improved methods, the minimum fuzziness is used as the fitting criterion, and linear programming is kept as the problem-solving technique.

The main shortcoming of the above method is that the concept of least-squares is not utilized; therefore, a natural extension of fuzzy regression would be the integration of the least-squares criterion into fuzzy regression as described in the next section.

3. Fuzzy least-squares regression

In this section, the development of fuzzy least-squares regression analysis (FLSRA) is described. Different aspects of fuzzy least-squares regression were investigated by Celminš [1,2], Diamond [6], Savic and Pedrycz [12] and Chang and Ayyub [3]. Celminš [1] defines a compatibility measure between fuzzy data and a model, and uses this measure as a model-fitting criterion. Diamond [6] developed a fuzzy least-squares method. Savic and Pedrycz [12] proposed a combined approach for FLSRA by integrating minimum fuzziness criterion into the ordinary least-squares regression. Chang and Ayyub [3] discussed reliability issues of FLSRA, such as standard error and correlation coefficient. In this section, two methods of FLSRA, as developed by

Celminš [1,2] and Savic and Pedrycz [12], are summarized.

3.1. Fuzzy least-squares regression using maximum compatibility criterion

Celminš [1] proposed an approach for fuzzy least-squares regression, based on a compatibility measure between data and a fitted model. Defining $\mu_{\tilde{A}}(X)$ and $\mu_{\tilde{B}}(X)$ as the membership functions of two fuzzy quantities $\tilde{A}$ and $\tilde{B}$, Celminš suggested a compatibility measure between $\tilde{A}$ and $\tilde{B}$ as $\gamma(\tilde{A},\tilde{B})$. For example, if $\mu_{\tilde{A}}(X)$ and $\mu_{\tilde{B}}(X)$ are two normalized triangular membership functions, then $\gamma(\tilde{A},\tilde{B})$ can be expressed as follows and as shown in Fig. 2(a):

$$\gamma(\tilde{A},\tilde{B}) = \max_{x} \min\{\mu_{\tilde{A}}(X), \mu_{\tilde{B}}(X)\}. \tag{5}$$

The value of γ is between zero and unity. Two extreme cases of the compatibility measure are: $\gamma = 0$, if the widths of two fuzzy quantities do not overlap as shown in Fig. 2(b), and $\gamma = 1$, if the centers of two fuzzy quantities overlap as shown in Fig. 2(c). The compatibility measure γ has a similar purpose as the degree of belief h in the previous section.

The objective of data fitting according to this approach is to find a model such that the overall compatibility between data and a fitted model is at its maximum. Let γ_i be the measure of compatibility between each datum and the fitted model, then a measure for the overall compatibility is the sum of squares of the deviations of γ_i from unity, i.e., the objective function for the data is to minimize the sum of squares of deviations (W) as

$$W = \sum_{i=1}^{m} (1-\gamma_i)^2. \tag{6}$$

According to this approach, the final formula for the fuzzy least-squares regression using maximum compatibility is as follows:

$$\hat{Y} = \tilde{A}_0 + \tilde{A}_1 X = m_0 + m_1 X \pm \sqrt{c_0^2 + 2c_{01}X + c_1^2 X^2}. \tag{7}$$

The first part of Eq. (7), $m_0 + m_1 X$, represents the centerline of fuzzy regression model. The coefficients m_0 and m_1 are obtained by a weighted least-squares regression, and the term $1/(\text{datum fuzziness})^2$ is used as the weight assigned for each datum. And, the second part of Eq. (7), $\pm\sqrt{c_0^2 + 2c_{01}X + c_1^2 X^2}$ specifies the upper and lower fuzzy outer boundary of the regression model. The c_0 and c_1 are the fuzzy half-widths of coefficients $\tilde{A}_0$ and $\tilde{A}_1$. According to Celminš' [1] definition the c_{01} is the fuzzy concordance between $\tilde{A}_0$ and $\tilde{A}_1$. The concordance between two fuzzy parameters has a similar meaning as the probabilistic covariance between two ordinary parameters. Through iterative computations, the c_0, c_1, and c_{01} are obtained by applying Eq. (7) with a desired compatibility measure between 0 and 1.

For crisp X and crisp Y data, the fuzzy regression problem was described in Celminš [2]. Because the method uses the same assumption of system fuzziness by Tanaka et al. [16], the resulting regression model is also a fuzzy equation as Eq. (7). According to this method, a fuzzy regression equation can fit all crisp data inside the fuzzy model, and provide a maximum compatibility. And, the difference between model fitting for crisp data and model fitting for fuzzy data is in the computational procedure and the numerical results.

3.2. Fuzzy least-squares regression using the minimum fuzziness criterion

Savic and Pedrycz [12] formulated the fuzzy regression method by combining the least-squares principle and minimum fuzziness criterion. The method is performed in two consecutive steps. The first step uses ordinary least-squares regression to find fuzzy center values of fuzzy regression coefficients. The second step uses the minimum fuzziness criterion to find the fuzzy widths of fuzzy regression coefficients.

In the first step, a regression line is fitted to the data using the available information about the center value of the fuzzy observations. The fuzzy data are treated as simplified crisp data, and the regression analysis is performed as a ordinary least-squares regression. The results of the first step are used as center values of the fuzzy regression coefficients.

In the second step, fuzzy coefficients are determined using the minimum fuzziness criterion. The widths of the fuzzy coefficients are determined by Eqs. (3)–(5)

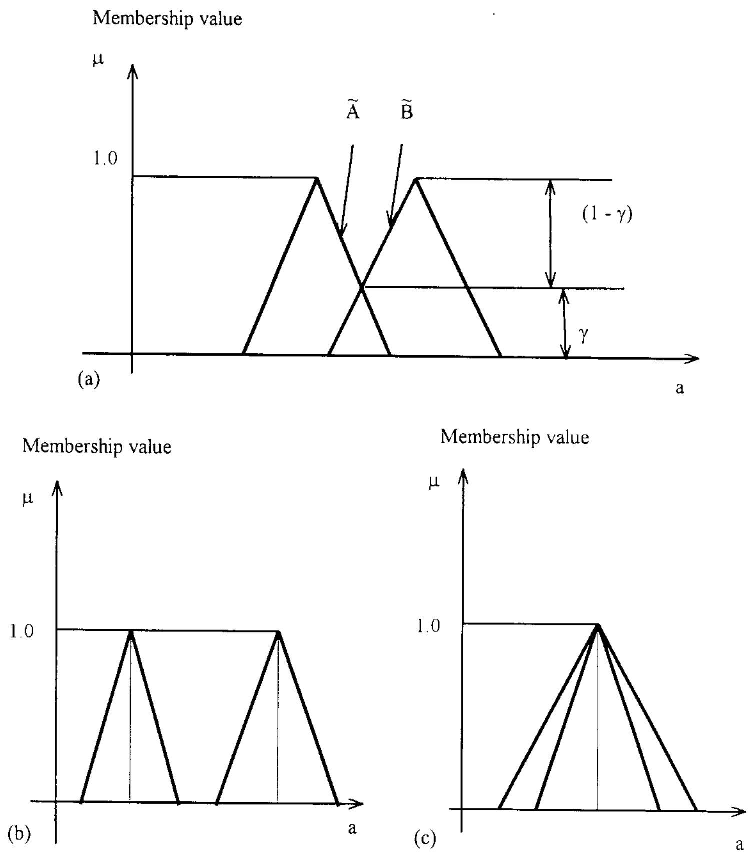

Fig. 2. Compatibility measure. (a) Definition of compatibility measure γ between two fuzzy quantities $\tilde{A}$ and $\tilde{B}$, (b) $\gamma = 0$, (c) $\gamma = 1$.

as the minimum fuzziness method with the difference of using the fuzzy centers of regression coefficients resulting from the first step.

4. Interval regression

According to this method, the fuzzy data and fuzzy regression coefficients are treated as interval numbers. The interval operations [9,10] are applied in fuzzy regression, therefore called interval regression analysis [8]. The fuzzy regression coefficients are determined such that all fuzzy outputs are within a fuzzy regression model. An interval regression for crisp X and crisp Y, is shown in Fig. 3(a). An interval regression model for crisp X and fuzzy Y, is shown in Fig. 3(b). For $\hat{Y} = \tilde{A}_0 + \tilde{A}_1 X$, the following linear programming formulation is used to solve for the fuzzy regression coefficients $\tilde{A}_0 = (m_0, c_0)$ and $\tilde{A}_1 = (m_1, c_1)$:

$$\text{minimize} \quad nc_0 + c_1 \sum_{i=1}^{n} X_i \tag{8}$$

subject to $c_0 \geqslant 0$, and $c_1 \geqslant 0$,

$$(m_0 - c_0) + (m_1 - c_1)X_i \leqslant Y_{i,\mathrm{L}} \quad \text{for } i = 1 \text{ to } n, \tag{9}$$

$$(m_0 + c_0) + (m_1 + c_1)X_i \geqslant Y_{i,\mathrm{U}} \quad \text{for } i = 1 \text{ to } n, \tag{10}$$

where $Y_{i,\mathrm{L}}$ and $Y_{i,\mathrm{U}}$ are the lower and upper limits for each fuzzy datum, respectively. The objective function of Eq. (8) results in the minimization of the total fuzzy widths. The constraints, Eqs. (9) and (10), are used to

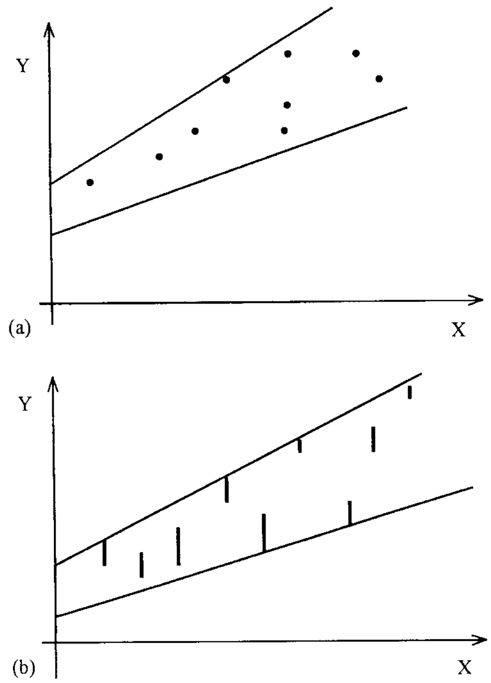

Fig. 3. Interval regression models. (a) Interval regression model for crisp X and crisp Y, (b) Interval regression model for crisp X and fuzzy Y.

confine all the observed fuzzy data within the fuzzy regression model.

The above formulation is called a minimization problem according to Ishibuchi [8]. Besides, a maximization problem for interval regression was also described in Ishibuchi [8]. To the contrary, the maximization problem is based on confining the estimated $\hat{Y}_i$ within the observed Y_i. Only the minimization problem is discussed in this section.

5. Numerical examples

Numerical examples are used in this section to illustrate the fuzzy regression models that are summarized in previous sections. The following data pairs of $(X_i : Y_i;\ i = 1, 2, \ldots, 8)$ are used to demonstrate the crisp X and crisp Y data case [8]:

$$[X_i : Y_i] = [(2:14), (4:16), (6:14), (8:18), (10:18), (12:22), (14:18), (16:22)]. \tag{11}$$

By adding ± 1.0 to Y_i, the above data are fuzzified to produce the case of crisp X and fuzzy $\tilde{Y}$ data case as follows:

$$[X_i, \tilde{Y}_i] = [(2:(14,1)), (4:(16,1)), (6:(14,1)), (8:(18,1)), (10:(18,1)), (12:(22,1)), (14:(18,1)), (16:(22,1))]. \tag{12}$$

The two data cases, Eqs. (11) and (12), are used in this section to demonstrate the fuzzy regression methods. The results of each fuzzy regression model is compared to the results of the ordinary least-squares regression. And, a comparative discussion of fuzzy regression methods is provided at the subsequent section.

5.1. Crisp X and crisp Y data case

For ordinary crisp X and crisp Y data, the following ordinary least-squares regression equation is used as a benchmark for the fuzzy regression models and the fuzzy least-squares regression models:

$$\hat{Y} = A_0 + A_1 X = 12.93 + 0.54X. \tag{13}$$

Besides the regression equation, reliability measures are also important for regression analysis. They are the standard deviation of observations $S_y = 3.11$, the standard error of estimate $S_e = 1.79$, and the correlation coefficient $R = 0.84$.

5.1.1. Fuzzy regression model using minimum fuzziness criterion

According to this method, each datum is translated into two constraints, Eqs. (3) and (4). Eight data pairs of Eq. (11) generate 16 constraints. The objective function, Eq. (2), is to minimize the total fuzzy widths of the regression coefficients. Therefore, the linear programming formulation of Eqs. (2)–(4), can be solved to obtain the fuzzy centers and the fuzzy half-widths of regression coefficient. Two cases of degree of belief, $h = 0.0$ and 0.70, were solved, and their fuzzy regression models are listed below.

(a) *For* $h = 0.0$:

$$\hat{Y} = \tilde{A}_0 + \tilde{A}_1 X = (12.00, 1.00) + (0.63, 0.13)X, \tag{14}$$

where (12.00, 1.00) is a fuzzy intercept with a center value of 12.00 and half-width of 1.00; and (0.63, 0.13) is a fuzzy slope with center value of 0.63 and half-width of 0.13.

(b) *For* $h = 0.70$:

$$\hat{Y} = \tilde{A}_0 + \tilde{A}_1 X = (12.00, 3.33) + (0.63, 0.42)X. \quad (15)$$

The results of Eqs. (14) and (15) were compared to the results of the ordinary least-squares regression of Eq. (13) by plotting them in Fig. 4. From the numerical results, several observations can be made. For different degrees of belief, the fuzzy center values remain unchanged; while the higher the degree of belief, the fuzzier the fuzzy regression model. Therefore, when h equals 0, the fuzzy regression model has the narrowest fuzzy widths among all h between 0.0 and 1.0. From Eqs. (14) and (15), the centerline equation of fuzzy models is $Y = 12.00 + 0.63X$, which is different from the ordinary least-squares regression equation, i.e., $Y = 12.93 + 0.54X$. Such difference is also shown in Fig. 4.

5.1.2. *Fuzzy least-squares regression model using maximum compatibility criterion*

According to this method, crisp data are fitted into a fuzzy regression model at a specified compatibility measure γ. The computation of FLSRA using maximum compatibility criterion is achieved by an iterative procedure. For comparison purposes, two values of compatibility measure $\gamma = 0.0$ and 0.7 were used, and the corresponding results of FLSRA are listed as below.

(a) $\gamma = 0.0$:

$$\hat{Y} = m_0 + m_1 X \pm \sqrt{c_0 + 2c_{01}X + c_1 X^2}$$
$$= 12.55 + 0.59X$$
$$\pm\sqrt{25.67 + 2(-2.49)X + 0.28X^2}. \quad (16)$$

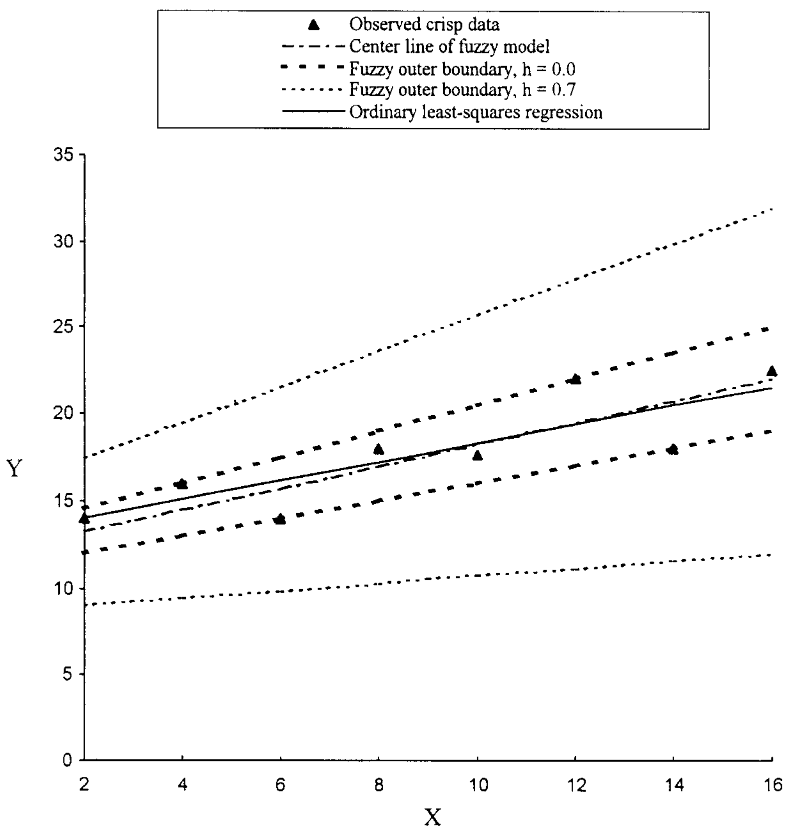

Fig. 4. Fuzzy regression model using minimum fuzziness criterion for crisp X and crisp Y.

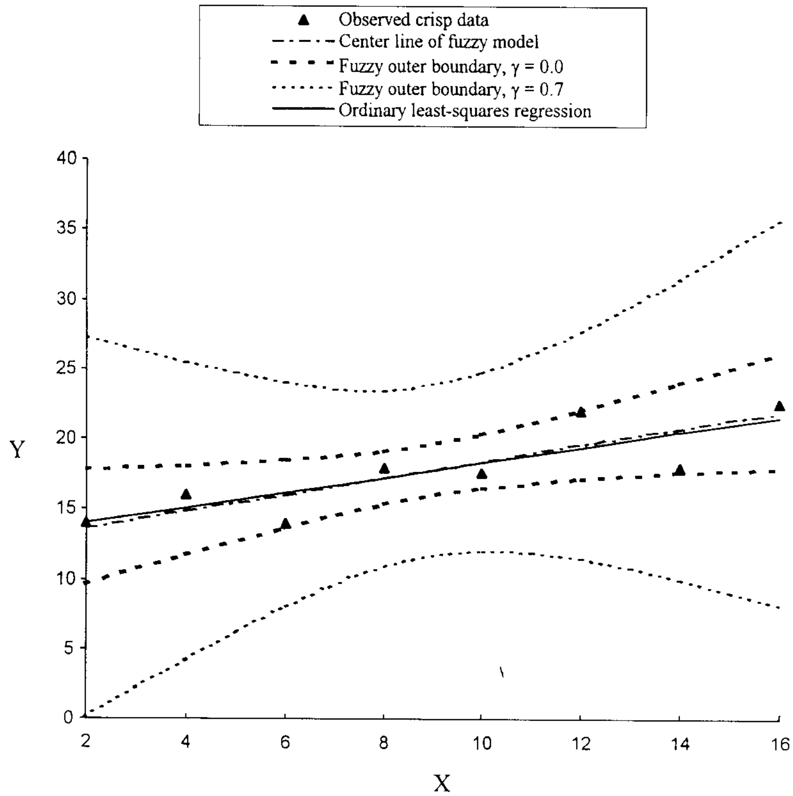

Fig. 5. Fuzzy least-squares regression model using maximum compatibility criterion for crisp X and crisp Y.

(b) $\gamma = 0.7$:

$$\hat{Y} = m_0 + m_1 X \pm \sqrt{c_0 + 2c_{01}X + c_1 X^2}$$
$$= 12.55 + 0.59X$$
$$\pm\sqrt{285.22 + 2(-27.68)X + 3.08X^2}. \qquad (17)$$

The results of Eqs. (16) and (17) include two parts. The first part of each formula is the centerline equation; and the second part of each formula is the two parabolic curves for upper bound and lower bound of the fuzzy regression model. The results of Eqs. (16) and (17) were compared with ordinary least-squares regression by plotting them in Fig. 5. From the numerical results, one can observe similar properties as for the method of minimum fuzziness. First, the centerline equation remains the same for different compatibility levels. The higher the desired compatibility level between data and model, the larger the fuzzy width. When $\gamma = 0.0$, the fuzzy regression model has the narrowest fuzzy width among all γ values between 0.0 and 1.0. Secondly, the centerline equation is different from the ordinary least-squares regression equation, i.e., Eq. (14). This is due to the fact that the maximum compatibility criterion minimizes the squares of (1 − compatibility (γ)). However, in ordinary least-squares regression, the fitting criterion is to minimize the sum of squares of (observed value − predicted value).

5.1.3. *Fuzzy least-squares regression model using minimum fuzziness criterion*

As was previously described, there are two steps in calculating fuzzy regression coefficients. First, the fuzzy centers are obtained from the ordinary least-squares regression. The results are the same as Eq. (13). Second, substituting these fuzzy centers into the linear programming problem of Eqs. (2)–(4), the fuzzy widths of regression coefficients can be determined. The two cases of $h = 0.0$ and 0.7 are shown

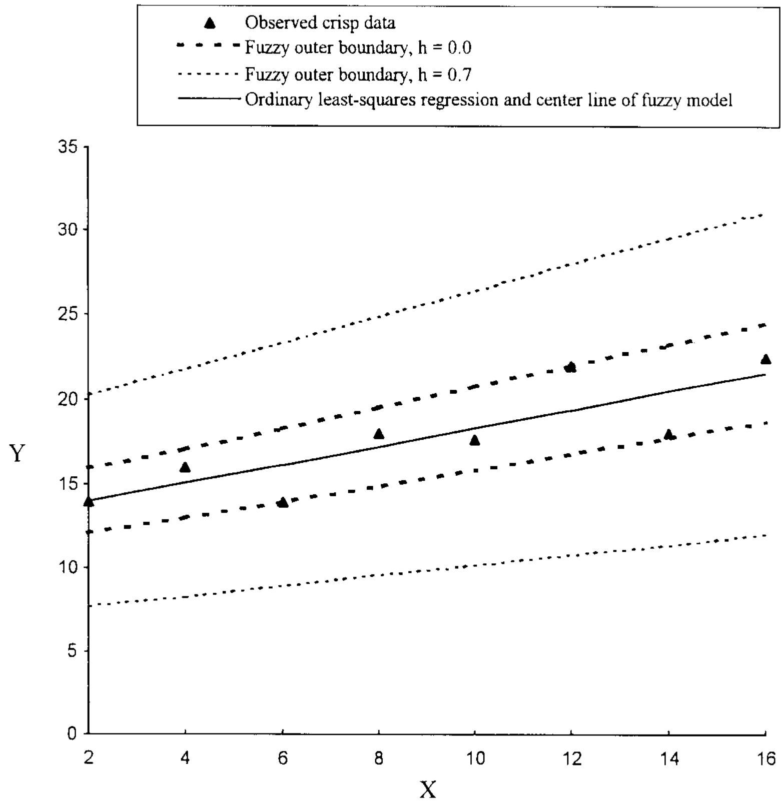

Fig. 6. Fuzzy least-squares regression model using minimum fuzziness criterion for crisp X and crisp Y.

herein.

(a) *For* $h = 0.0$:

$$\hat{Y} = \tilde{A}_0 + \tilde{A}_1 X = (12.93, 1.75) + (0.54, 0.07)X. \quad (18)$$

(b) *For* $h = 0.70$:

$$\hat{Y} = \tilde{A}_0 + \tilde{A}_1 X = (12.93, 5.83) + (0.54, 0.23)X. \quad (19)$$

The results of Eqs. (18) and (19) were compared to the results of ordinary least-squares regression of Eq. (13) by plotting them in Fig. 6. From Eqs. (18) and (19), the centerline equation of fuzzy models is the same as the ordinary least-squares regression equation, i.e., $Y = 12.93 + 0.54X$. Since the minimum fuzziness criterion is used in this method, Eqs. (18) and (19) also show the same property as Eqs. (16) and (17), i.e., the higher the degree of belief h, the fuzzier the regression model. However, in any case, the results of the fuzzy least-squares regression do not approach the results of the ordinary least-squares regression as h approaches 0. Intuitively, the fuzzy least-squares regression should produce the same results of the ordinary least-squares regression, as h approaches 0.

5.1.4. *Interval regression model*

The objective of the interval regression method is to find the minimum width model that contains all data points. Applying Eqs. (8)–(10) to the crisp X and crisp Y data example, the following interval regression model was obtained:

$$\hat{Y} = \tilde{A}_0 + \tilde{A}_1 X = (12.00, 1.00) + (0.63, 0.13)X. \quad (20)$$

This interval regression model was compared to the ordinary least-squares regression model of Eq. (13) by plotting them in Fig. 7. It can be observed that all data points are included within the interval regression model. Also, Eq. (22) is identical to Eq. (16), the

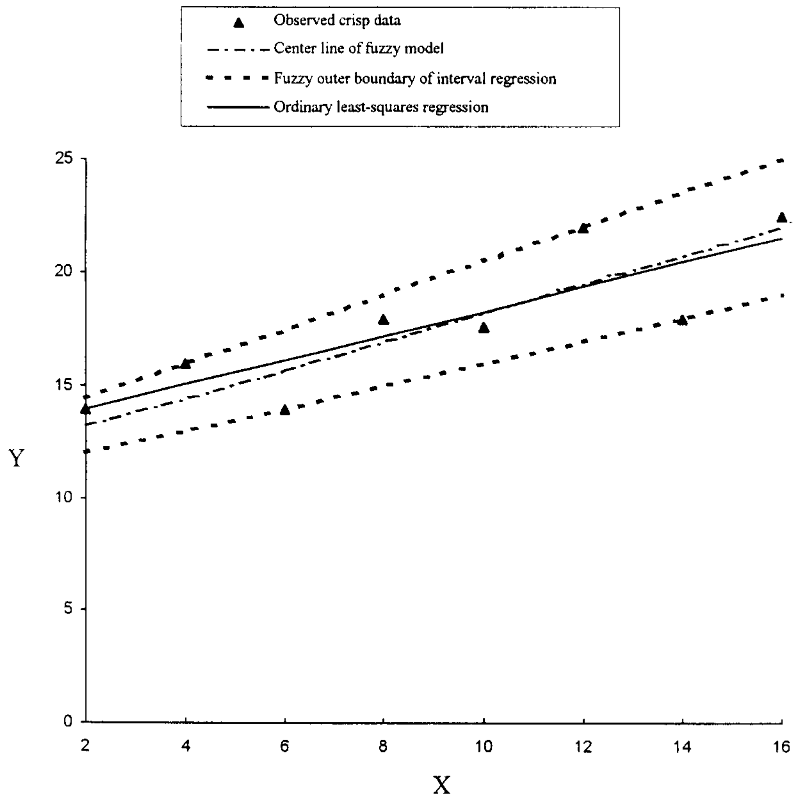

Fig. 7. Interval regression model for crisp X and crisp Y.

fuzzy regression equation using the minimum fuzziness criterion with $h = 0.0$.

5.2. Crisp X and fuzzy Y data case

Since the ordinary least-squares regression only applies to crisp data, the crisp X and fuzzy Y data can be analyzed by only fuzzy regression and fuzzy least-squares regression methods. However, Eq. (16) of the ordinary least-squares regression is still used for comparison purposes.

5.2.1. Fuzzy regression model using minimum fuzziness criterion

According to this method, the linear programming problem of Eqs. (3)–(5) is solved to obtain the fuzzy centers and the fuzzy widths of regression coefficients. Also, Eqs. (4) and (5) provide a means to include fuzzy Y data, in terms of h. Two cases of $h = 0.01$ and 0.70 were solved, and their results are reported herein.

(a) *For* $h = 0.0$:

$$\hat{Y} = \tilde{A}_0 + \tilde{A}_1 X = (12.00, 2.00) + (0.63, 0.13)X. \quad (21)$$

(b) *For* $h = 0.70$:

$$\hat{Y} = \tilde{A}_0 + \tilde{A}_1 X = (12.00, 4.33) + (0.63, 0.42)X. \quad (22)$$

The results of Eqs. (21) and (22) were compared to the results of ordinary least-squares regression of Eq. (13) by plotting them in Fig. 8. The fuzzy regression model has similar properties to the example of crisp X and crisp Y. For different degrees of belief h, the fuzzy center values remain the same. The higher the degree of belief h, the fuzzier the fuzzy regression model. When h approaches 0, the fuzzy regression model has the narrowest fuzzy widths among all h values between 0 and 1.

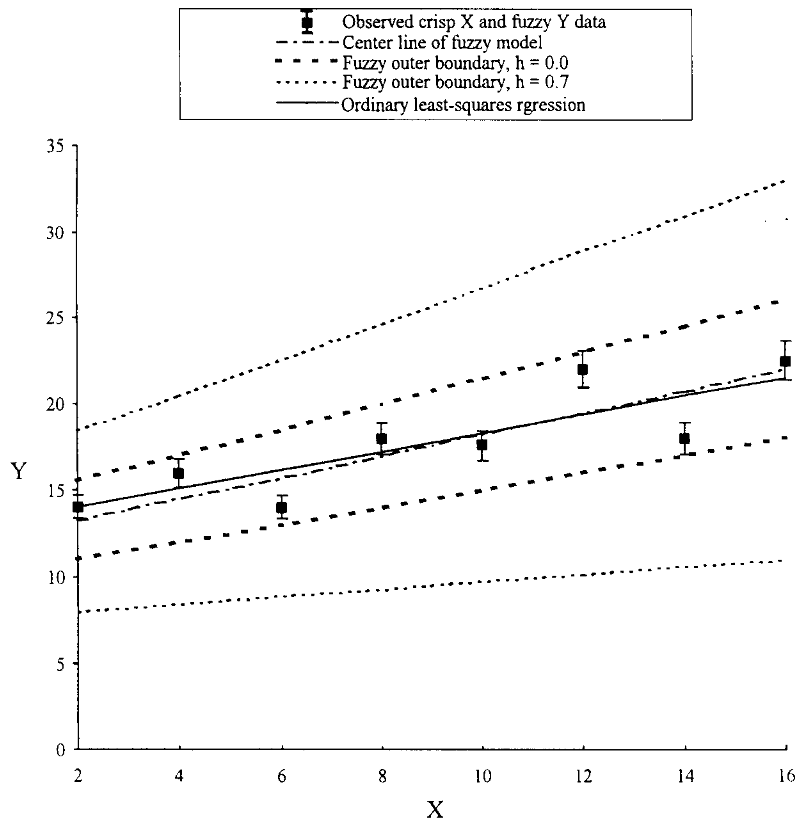

Fig. 8. Fuzzy regression model using minimum fuzziness criterion for crisp X and fuzzy Y.

5.2.2. Fuzzy least-squares regression model using maximum compatibility criterion

Two cases of compatibility measures $\gamma = 0.0$ and 0.7 were solved, and their results of FLSRA are listed below.

(a) $\gamma = 0.0$:

$$\begin{aligned}\hat{Y} &= m_0 + m_1X \pm \sqrt{c_0^2 + 2c_{01}X + c_1^2X^2} \\ &= 12.93 + 0.54X \\ &\quad \pm\sqrt{9.18 + 2(-0.81)X + 0.09X^2}. \qquad (23)\end{aligned}$$

(b) $\gamma = 0.7$:

$$\begin{aligned}\hat{Y} &= m_0 + m_1X \pm \sqrt{c_0^2 + 2c_{01}X + c_1^2X^2} \\ &= 12.93 + 0.54X \\ &\quad \pm\sqrt{207.36 + 2(-18.30)X + 2.03X^2}. \qquad (24)\end{aligned}$$

The results of Eqs. (23) and (24) were compared to the results of the ordinary least-squares regression of Eq. (13) by plotting them in Fig. 9. From the numerical results, several observations can be made. First, the centerline equation is identical to the ordinary least-squares regression equation of Eq. (13). Second, the higher the desired compatibility level, the wider the fuzzy width. When $\gamma = 0.0$, the fuzzy regression model has the narrowest fuzzy width among all γ between 0.0 and 1.0.

It is noted that the centerline equations of Eqs. (23) and (24) are the same as the ordinary least-squares regression equation, i.e., $Y = 12.93 + 0.54X$. According to Celminš [1,2] and the discussion of Section 3.1, for fuzzy data, Celminš uses the ordinary least-squares criterion for the centerline equation and the maximum compatibility criterion for the fuzzy range. Therefore, the centerline equation of fuzzy model is the same as the ordinary least-squares regression equation.

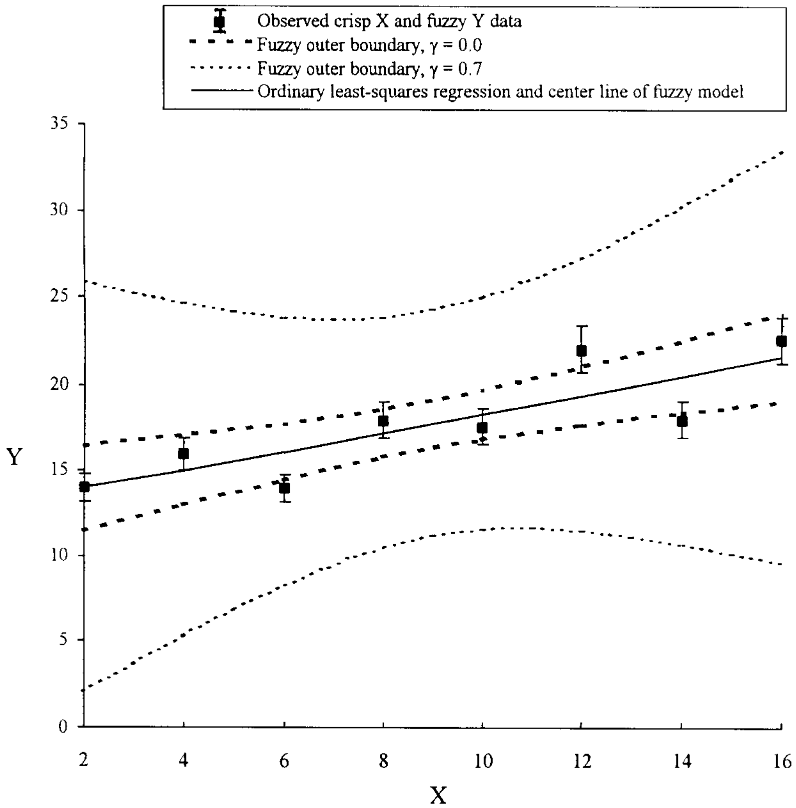

Fig. 9. Fuzzy least-squares regression model using maximum compatibility criterion for crisp X and fuzzy Y.

5.2.3. Fuzzy least-squares regression model using minimum fuzziness criterion

In this case, the middle values of fuzzy regression coefficients are determined by simplifying the $\tilde{Y}$ fuzzy values as crisp numbers, and solved by the ordinary least-squares regression. The results are certainly the same as Eq. (13). Then, the fuzzy half widths of regression coefficients are determined by solving the linear programming problem of Eqs. (2)–(4). The following two cases of $h = 0.0$ and 0.7 were used, and their fuzzy least-squares regression models are listed below.

(a) *For* $h = 0.0$:

$$\hat{Y} = \tilde{A}_0 + \tilde{A}_1 X = (12.93, 1.07) + (0.54, 0.21)X. \quad (25)$$

(b) *For* $h = 0.7$:

$$\hat{Y} = \tilde{A}_0 + \tilde{A}_1 X = (12.93, 2.833) + (0.54, 0.567)X. \quad (26)$$

The results of Eqs. (25) and (26) were compared to the results of ordinary least-squares regression of Eq. (13) by plotting them in Fig. 10. In this method, the center values of the fuzzy observed data, which are crisp numbers, are used to obtain the fuzzy centers of the regression coefficients. Since the least-squares criterion is used, the results are identical to the ordinary least-squares regression. Also, since the minimum fuzziness criterion is used, the fuzzy models get fuzzier as h approaches 1.0.

5.2.4. Interval regression model

Applying Eqs. (8)–(10) to the crisp X and fuzzy Y data example of Eq. (12), the following interval regression equation was obtained:

$$\hat{Y} = \tilde{A}_0 + \tilde{A}_1 X = (12.00, 2.00) + (0.63, 0.13)X. \quad (27)$$

The results of Eq. (27) in comparison with ordinary least-squares regression of Eq. (13) were plotted in

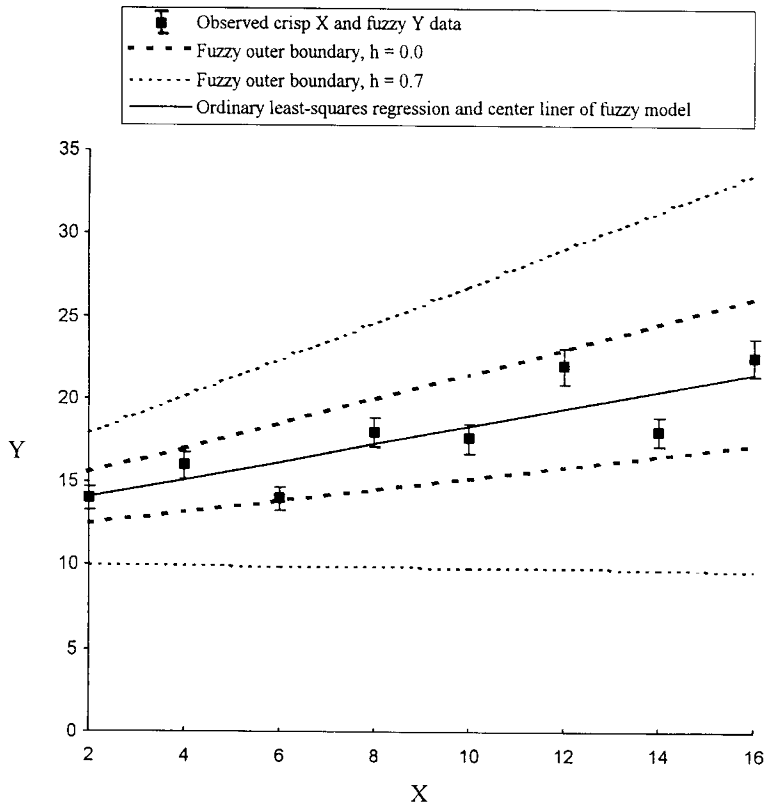

Fig. 10. Fuzzy least-squares regression model using minimum fuzziness criterion for crisp X and fuzzy Y.

Fig. 11. It can be observed that all fuzzy data fall within the interval regression model. Also, Eq. (27) is identical to Eq. (21), the fuzzy regression equation using the minimum fuzziness criterion with $h = 0.0$.

6. Summary and discussion

A comprehensive literature review on fuzzy regression is provided in this paper. As a result of the review, four fuzzy regression methods are described and illustrated using numerical examples. Both crisp X and crisp Y data, and crisp X and fuzzy Y data are used to demonstrate each fuzzy regression method. For comparison purposes, the fuzzy regression equations using crisp X and crisp Y data are listed in Table 1. The fuzzy regression equations using crisp X and fuzzy Y data are listed in Table 2. Based on the comparison, their results are summarized and discussed in this section.

Except for the method of FLSRA using maximum compatibility criterion, all other fuzzy regression methods use linear programming (LP) in estimating the fuzzy coefficients in the resulting models. For simplicity, only eight sets of data (sample sizes) are used in the numerical examples. As the number of data sets increases, the following two difficulties in using linear programming (LP) to estimate fuzzy coefficients can result:

1. *Linear programming formulation*: Each data set results in two constraints in the fuzzy regression formulation. As the number of data sets increase, the number of constraints increases proportionally. This increase might result in computational difficulties using LP software or computers. Also, when adding to or removing from the independent variables, the whole set of constraints must be reformulated. The resulting inconvenience might limit experimenting with variables to obtain the optimum number of independent variables in regression analysis. Therefore,

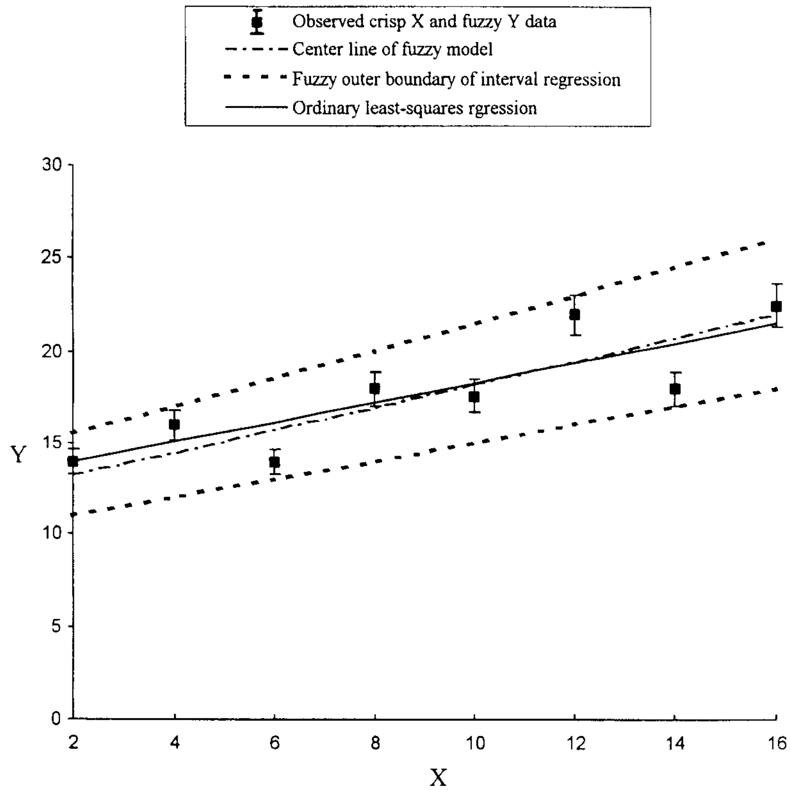

Fig. 11. Interval regression model for crisp X and fuzzy Y.

Table 1
Fuzzy regression equations using crisp X and crisp Y data

Fuzzy regression method	Regression equation
A. Fuzzy regression using the minimum fuzziness criterion at $h=0.0$	$\hat{Y}=(12.00, 1.00)+(0.63, 0.13)X$
B. Fuzzy regression using the minimum fuzziness criterion at $h=0.7$	$\hat{Y}=(12.00, 3.33)+(0.63, 0.42)X$
C. Fuzzy least-squares regression using the maximum compatibility criterion at $\gamma=0.0$	$\hat{Y}=12.55+0.59X \pm\sqrt{2.567+2(-2.49)X+0.28X^2}$
D. Fuzzy least-squares regression using the maximum compatibility criterion at $\gamma=0.7$	$\hat{Y}=12.55+0.59X \pm\sqrt{285.22+2(-27.68)X+3.08X^2}$
E. Fuzzy least-squares regression using the minimum fuzziness criterion at $h=0.0$	$\hat{Y}=(12.93, 1.75)+(0.54, 0.07)X$
F. Fuzzy least-squares regression using the minimum fuzziness criterion at $h=0.7$	$\hat{Y}=(12.93, 5.83)+(0.54, 0.23)X$
G. Interval regression	$\hat{Y}=(12.00, 1.00)+(0.63, 0.13)X$
H. Ordinary least-squares regression	$\hat{Y}=12.93+0.54X$

Table 2
Fuzzy regression equations using crisp X and fuzzy Y data

Fuzzy regression method	Regression equation
A. Fuzzy regression using the minimum fuzziness criterion at $h = 0.0$	$\hat{Y} = (12.00, 2.00) + (0.63, 0.13)X$
B. Fuzzy regression using the minimum fuzziness criterion at $h = 0.7$	$\hat{Y} = (12.00, 4.33) + (0.63, 0.42)X$
C. Fuzzy least-squares regression using the maximum compatibility criterion at $\gamma = 0.0$	$\hat{Y} = 12.93 + 0.54X \pm \sqrt{9.18 + 2(-0.81)X + 0.09X^2}$
D. Fuzzy least-squares regression using the maximum compatibility criterion at $\gamma = 0.7$	$\hat{Y} = 12.93 + 0.54X \pm \sqrt{207.36 + 2(-18.30)X + 2.03X^2}$
E. Fuzzy least-squares regression using the minimum fuzziness criterion at $h = 0.0$	$\hat{Y} = (12.93, 1.07) + (0.54, 0.21)X$
F. Fuzzy least-squares regression using the minimum fuzziness criterion at $h = 0.7$	$\hat{Y} = (12.93, 1.07) + (0.54, 0.21)X$
G. Interval regression	$\hat{Y} = (12.00, 2.00) + (0.63, 0.13)X$
H. Ordinary least-squares regression	$\hat{Y} = 12.93 + 0.54X$

the formulation of linear programming can limit the use of fuzzy regression.

2. *Sign of unknown variables*: The LP formulation restricts the sign of unknown variables to be nonnegative. The nonnegative condition is a requirement in solution procedures of the LP formulation. For a fuzzy coefficient $\tilde{A} = (m, c)$, while the fuzzy width c remains positive or zero, the fuzzy center can be positive or negative. Unless the explicit influence of an independent variable to the outcome is known, fuzzy coefficients can be positive or negative. Therefore, the unknown fuzzy center values have to be presented as a linear combination of two nonnegative variables. That is, an unknown variable m is presented as $m = m^+ - m^-$, where m^+ and m^- are both nonnegative variables. The sign of the unknown variable is then determined by the difference of two nonnegative numbers. The nonnegativity requirement increases the number of unknown variables. This creates another problem for the LP formulation, particularly when the number of data sets is large.

Besides the possible difficulties in using the LP formulation, the limiting behavior of the fuzzy regression methods needs to be examined. The limiting behavior is defined as when fuzziness is decreased in the fuzzy regression, the results of the fuzzy regression should approach the results of the ordinary regression. However, the minimum fuzziness criterion and the maximum compatibility criterion generate a fuzzy regression model, where the higher the h (or γ) value, the larger the fuzzy width. When h (or γ) equals 1.0, the fuzzy regression models have the largest fuzzy width. In any case, the fuzzy regression models do not approach the ordinary least-squares regression models as h (or γ) approaches its limiting values. The lack of this property has unfortunately segregated the use of fuzzy regression from the well-received ordinary least-squares regression. For the same reason, the use of fuzzy regression methods has drawn some criticism from statisticians, for example, [11]. Nevertheless, it is our belief that fuzzy regression and ordinary regression should be integrated rather than segregated. The segregation is a limitation to the fuzzy regression methods.

In interval regression analysis, Eq. (20) is the same as Eq. (14) for crisp data, and Eq. (27) is the same as Eq. (21) for fuzzy data. Therefore, the interval regression analysis is a special case of the minimum fuzziness method when h equals zero. However, the interval regression analysis presents a new concept for the fitting criterion, i.e., confining all fuzzy data within a fuzzy regression model. Tanaka and Ishibuchi [14] used the same fitting criterion for nonlinear fuzzy regression by a neural networks technique.

After performing regression analysis, there is a need to evaluate reliability measures for the corresponding regression equation. Reliability measures for fuzzy regression are discussed by Chang and Ayyub [3].

7. Conclusions

According to the numerical examples and the discussion, fuzzy regression methods are based on a system fuzziness assumption, and presents different results from ordinary regression. The contrast between fuzzy regression and ordinary regression has been due to the difference in views on the meaning of deviations between observed values and estimated values. In ordinary regression, the deviations are viewed as *random errors* due to observation inconsistency. In fuzzy regression, the deviations are viewed as *fuzzy errors* due to system fuzziness. Both fuzzy regression and ordinary regression only consider part of the totality of uncertainty. In fact, randomness and fuzziness are two different kinds of uncertainty, co-existing in a regression analysis. In ordinary regression analysis, probability theory is used to model random errors, and the result is presented as a ordinary regression equation. On the other hand, fuzzy set theory can be used to model fuzzy errors, and the result can be presented using a fuzzy regression equation.

Ordinary regression analysis provides a suitable tool for dealing with crisp observed data by analyzing the random errors between estimated values and observed values. In ordinary regression, crisp data are considered to include random errors, which are different from fuzzy errors in fuzzy data. Fuzzy regression analysis provides a way to model observed fuzzy data, such as linguistic descriptions of the type: excellent, very good, and good. If data contain fuzziness, then fuzzy regression needs be used. However, when the data are made to approach a crisp state in fuzzy regression analysis, the results of fuzzy regression should approach the results of ordinary regression. But, such a property does not exist in currently used fuzzy regression methods. The reason is that fuzzy regression uses the system fuzziness assumption as a replacement of the randomness assumption in ordinary regression. The fuzziness of the data is treated as a substitute to the randomness in data, rather than an addition to the randomness in the data. Based on this assumption, fuzzy regression models are the results of data fuzziness and system fuzziness. Therefore, different fitting criteria need to be used for a fuzzy regression model. Consequently, even when all data are crisp, fuzzy regression can still provide a fuzzy model containing the assumed system fuzziness [16].

In conclusion, randomness and fuzziness are two different kinds of uncertainty that co-exist in a regression analysis. Both randomness and fuzziness should add up to the total uncertainty in a regression model. A complete regression analysis should include both random and fuzzy types of uncertainty. But, fuzziness type of uncertainty exists only when regression data contain fuzziness.

In order to include both randomness and fuzziness into a regression analysis, a need for developing new regression method exists. Therefore, a new concept of hybrid regression analysis is proposed and developed in the accompanying paper. The reliability measures of fuzzy regression and hybrid regression are also in need of further development.

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