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Structural Analysis with Fuzzy Variables

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Abstract: *The analysis of existing structures requires engineers to model two types of uncertainty, cognitive and noncognitive. The objective of this paper is to reexamine structural analysis methods by considering the cognitive type of uncertainty. Two analytical approaches are proposed for this purpose: (1) combining the displacement method with fuzzy arithmetic and (2) considering all possible permutations of extreme values of any uncertain variables in a structure using the displacement method. The first approach, which is based on fuzzy arithmetic, requires less computing time as compared with the permutations method but only obtains approximate solutions. However, the second approach produces the exact solution. For the purpose of illustration, the modulus of elasticity E is assumed to be an uncertain variable and is modeled as a triangular fuzzy number. The structural behavior was investigated due to this cognitive uncertainty in E. The results based on the second approach show that if E is a triangular fuzzy number, the member forces can be either fuzzy numbers or crisp values, depending on the structural type. In addition, modified definitions for fuzzy division and fuzzy subtraction are proposed in this paper. Applications of these modified definitions and proposed methods are also presented.*

1. INTRODUCTION

Uncertainties in structural engineering systems can be attributed mainly to ambiguity and vagueness in defining the variables of the systems and their relations. The ambiguity component is generally due to noncognitive sources. These sources include (1) physical randomness; (2) statistical uncertainty due to the use of limited information in estimating the characteristics of these variables; and (3) model uncertainties that are due to simplifying assumptions in analytical and predicative models, simplified methods, and idealized representations of real performances. The vagueness-related uncertainty is due to cognitive sources, which include (1) the definition of some variables, e.g., structural performance (failure or survival), quality, deterioration, skill and experiences of construction workers and engineers, conditions of existing structures; (2) human error and other human factors; and (3) defining the interrelationships among the variables of problems, especially for complex systems.

1.1 Noncognitive uncertainty

Structural engineers and researchers have dealt with the ambiguity types of uncertainty in predicting structural behavior and in designing structural systems using the theories of probability and statistics. Probability distributions were used to model system variables that are uncertain. Probabilistic structural methods that include structural reliability methods, probabilistic engineering mechanics, stochastic finite-element methods, reliability-based design formats, random vibration, and other methods were developed and used for this purpose. In this treatment, however, probabilities also were used to deal with subjective information. Uniform and triangular probability distributions were frequently used to model this type of uncertainty for some variables. The Bayesian techniques also were used, for example, to deal with gaining information about these variables. The underlying distributions and probabilities were therefore updated.

1.2 Cognitive uncertainty

The cognitive uncertainty arises from mind-based abstractions of reality. These abstractions are therefore subjective and might lack crispness. This vagueness is distinct from ambiguity in source and natural properties, as mentioned before. The axioms of probability and statistics are limiting for the proper modeling and analysis of this uncertainty type and are not completely relevant or completely applicable. The modeling and analysis of the vagueness type of uncertainty in other civil engineering systems is discussed along

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with applications of fuzzy set theory to such systems in Ayyub.[1,2]

Since the inception of fuzzy set theory,[13-18] it has been used by scientists, researchers, and engineers in many fields. Information about these applications is provided, for example, in Kaufmann[8] and Kaufmann and Gupta.[9] Applications of fuzzy sets in civil engineering began in the early 1970s (e.g., Brown[4]). To date, many applications of the theory in engineering have been developed. The theory has been used successfully in, for example, (1) strength assessment of existing structures and other structural engineering applications; (2) risk analysis and assessment in engineering; (3) analysis of construction failures, scheduling of construction activities, safety assessment of construction activities, decisions during construction, and tender evaluations; (4) the impact assessment of engineering projects on the quality of wildlife habitat; (5) planning of river basins; (6) control of engineering systems; (7) computer vision; and (8) optimization based on soft constraints. These applications are described in Ayyub.[2]

The objective of this paper is to reexamine structural analysis methods by considering the cognitive type of uncertainty in structural engineering.

2 STRUCTURAL VARIABLES AND THEIR UNCERTAINTIES

Structural analysis requires the definition of some basic variables, such as geometry and dimensions, material properties, boundary conditions, loads, and methods of modeling and analysis. These basic variables can be uncertain in different forms and in varying amounts. Some of the variables might have noncognitive uncertainty, cognitive uncertainty, or both. For example, for an existing reinforced concrete structure, the modulus of elasticity E might be unknown. The modulus of elasticity can be estimated based on any available design or construction documents and judgment. The resulting estimate can be expressed in the form of a function that reflects the degree of belief of attaining certain values of E.

Researchers have used stochastic finite-element analysis (e.g., Contreras[5]) to deal with uncertain variables regardless of the uncertainty type. Therefore, this treatment might not reflect the real nature of uncertainty. Stochastic finite-element analysis is surely an adequate method to deal with the noncognitive uncertainty type but not the cognitive type. In the analysis of existing structures, both types of uncertainties exist and need to be properly considered. As a result, the following methods of structural analyses generally can be used:

1. Deterministic structural analysis for cases without uncertainty in basic variables;
2. Stochastic structural analysis for cases with noncognitive type of uncertainty (e.g., Contreras[5] and Vanmarcke and Grigoriu[11]);
3. Fuzzy structural analysis for cases with cognitive type of uncertainty;
4. Fuzzy-stochastic structural analysis for cases with both cognitive and noncognitive types of uncertainty.

This paper deals with the third case. The fourth case can be developed by combining cases 2 and 3.

3 FUZZY SETS FOR MODELING COGNITIVE UNCERTAINTY

As mentioned before, the modulus of elasticity E can be an uncertain variable that can be defined as a fuzzy number using fuzzy sets. A *fuzzy set* is a set whose boundary is not sharp and can be characterized by a membership grade function. The membership function $\mu_A(x)$ of a fuzzy set A can be defined as

$$\mu_A(x): X \rightarrow [0,1] \tag{1}$$

where X = a universal set. The membership function $\mu_A(x)$ for the fuzzy set A maps x from X to the real range $[0,1]$, which is the domain of $\mu_A(x)$. The membership function is the grade of compatibility of x in A or the degree of belief of x in A. If the shape of the membership function is triangular, the fuzzy set is called a *triangular fuzzy number*. An alpha-cut of a fuzzy set A can be defined as

$$A_\alpha = \{x \in X | \mu_A(x) \geq \alpha \qquad \text{for } \alpha \in [0,1]\} \tag{2}$$

For example, a triangular fuzzy number is shown in Fig. 1 with the following membership function $\mu_A(x)$:

$$\mu_A(x) = \begin{cases} 0 & x \leq 2 \\ x - 2 & 2 \leq x \leq 3 \\ 4 - x & 3 \leq x \leq 4 \\ 0 & x \geq 4 \end{cases} \tag{3}$$

The α-cut of A can be derived as

$$A_\alpha = [\alpha + 2, 4 - \alpha] \tag{4}$$

Using α = 1, 0.6, and 0 produces the following α-cuts of A, respectively:

$$A_{\alpha=1} = [3,3] \tag{5a}$$

$$A_{\alpha=0.6} = [2.6,3.4] \tag{5b}$$

$$A_{\alpha=0} = [2,4] \tag{5c}$$

A triangular fuzzy number can be expressed in the form of a triplet as follows:

$$A = [a,b,c] \tag{6}$$

where a = the lower x value of the triangle at $\alpha = 0$, b = the x value that corresponds to $\alpha = 0$, and c = the upper x

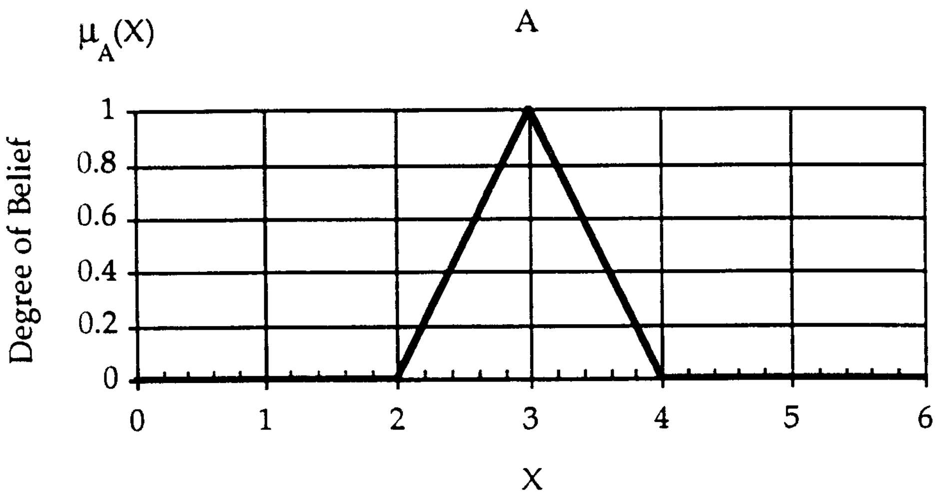


Fig. 1. Example of triangular fuzzy number.

value of the triangle at $\alpha = 0$. For example, A in Fig. 1 can be expressed as

$$A = [2,3,4] \tag{7}$$

In this paper, the modulus of elasticity E is assumed to be a triangular fuzzy number. For the purpose of illustration, the structural behavior due to this uncertainty in the modulus of elasticity E is investigated. The investigation is limited to one uncertain variable. However, the method can be generalized to simultaneously deal with several uncertain variables.

4 PROPOSED METHODS

There are two primary matrix methods of structural analysis based on the finite-element representation: (1) the force method (or the flexibility method) and (2) the displacement method (or the stiffness method). The displacement method has some advantages over the force method in structural analysis, especially in the application of digital computers (e.g., Hsieh[7]). Therefore, in this research, the displacement method is employed for structural analysis and the implementation of computer codes. Details about these matrix methods of structural analysis can be found in Beaufait et al.,[3] Hsieh,[7] and Weaver and Gere.[12]

According to the displacement method, nodal displacements are treated as the basic unknowns. The relationship between the nodal displacement vector $\{d\}$ and the corresponding nodal force vector $\{F\}$ can be expressed as

$$[K]\{d\} = \{F\} \tag{8}$$

where $[K]$ = the total stiffness matrix. Each nonzero element in the matrix $[K]$ contains the modulus of elasticity E. For crisp (without uncertainty) values of E, the vector $\{d\}$ can be solved directly by the Gaussian elimination method (e.g., Beaufait et al.[3]). For fuzzy values of E, the coefficients in $[K]$ are fuzzy numbers. Therefore, the nodal displacement vector $\{d\}$ and the member force vector are expected to become fuzzy vectors; i.e., each value in these vectors is a fuzzy number.

Two analytical approaches are presented to show the effects of uncertainty in E on the nodal displacements and member forces. The first approach combines the displacement method with fuzzy arithmetic. The second approach is based on considering all possible permutations of extreme values of E in a structure using the displacement method.

4.1 Analysis based on fuzzy arithmetic

This section introduces the fuzzy arithmetic used in this paper. Also, two modified definitions of fuzzy division and fuzzy subtraction are suggested.

For two fuzzy numbers A and B, let $A_\alpha = [a,b]$ and $B_\alpha = [c,d]$, where a, b, c, and d are real numbers and $\alpha \in [0,1]$. Fuzzy arithmetic (addition, subtraction, multiplication, and division, respectively) can be defined as follows[9,10]:

$$A_\alpha + B_\alpha = [a, b] + [c, d] = [a + c, b + d] \tag{9a}$$

$$A_\alpha - B_\alpha = [a, b] - [c, d] = [a - d, b - c] \tag{9b}$$

$$A_\alpha \times B_\alpha = [a, b] \times [c, d]$$
$$= [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)] \tag{9c}$$

$$A_\alpha / B_\alpha \; [a, b]/[c, d]$$
$$= [\min(a/c, a/d, b/c, b/d), \max(a/c, a/d, b/c, b/d)] \tag{9d}$$

Equation (9d) requires that $0 \notin [c, d]$. For example, assuming that $A_\alpha = [-3, 2]$ and $B_\alpha = [4, 5]$, the results from Eqs. (9) are

$$A_\alpha + B_\alpha = [-3, 2] + [4, 5]$$
$$= [-3 + 4, 2 + 5] = [1, 7] \quad (10a)$$
$$A_\alpha - B_\alpha = [-3, 2] - [4, 5]$$
$$= [-3 - 5, 2 - 4] = [-8, -2] \quad (10b)$$
$$A_\alpha \times B_\alpha = [-3, 2] \times [4, 5]$$
$$= [\min(-12, -15, 8, 10), \max(-12, -15, 8, 10)]$$
$$= [-15, 10] \quad (10c)$$
$$A_\alpha / B_\alpha = [-3, 2]/[4, 5]$$
$$= [\min(-3/4, -3/5, 2/4, 2/5), \max(-3/4, -3/5, 2/4, 2/5)]$$
$$= [-3/4, 2/4] \quad (10d)$$

The details of fuzzy arithmetic are provided in Kaufmann and Gupta.[9] Fuzzy arithmetic using alpha-cuts also can be viewed as interval arithmetic according to Moore,[10] since each alpha-cut of a triangular fuzzy number is an interval or a range.

In real (nonfuzzy) arithmetic, if A is a real number, then $A/A = 1$ for $A \neq 0$, and $A - A = 0$. In fuzzy arithmetic, if A is a fuzzy number, say $A_\alpha = [4,5]$, then by the definitions in Eqs. (9b) and (9d),

$$A_\alpha - A_\alpha = [4,5] - [4,5] = [-1,1] \quad (11a)$$

and

$$A_\alpha/A_\alpha = [4,5]/[4,5] = [4/5,5/4] \quad (11b)$$

The results according to Eqs. (11a) and (11b) are fuzzy numbers that contain 0 and 1, respectively. In engineering applications, it can be desirable to have crisp values of $A_\alpha - A_\alpha$ and A_α/A_α, i.e., values of crisp 0 and 1, respectively.

For example, consider a job that has two shifts every day, the morning shift and afternoon shift. The morning shift can be from 8:00 to 12:00 and the afternoon shift can be from 13:00 to 17:00. Every employee has to work at least 1 hour and at most 4 hours on each shift. If Mary works two shifts every day, then the ratio of the working time of the morning shift to that of the afternoon shift can be computed for a shift $A = [1,4]$ as

$$\text{The ratio} = \frac{A(\textit{morning})}{A(\textit{afternoon})} = \frac{[1, 4]}{[1, 4]}$$
$$= [1/4, 4] = [0.25, 4] \quad (12a)$$

The resulting ratio according to Eq. (12a) is uncertain. Now consider a case where $A = [1, 4]$ with the condition that Mary works the same hours on both shifts: therefore,

$$\text{The ratio} = \frac{A(\textit{morning})}{A(\textit{afternoon})} = \frac{[1, 4]}{[1, 4]} \text{ must be } 1 \quad (12b)$$

This example shows that the first answer (Eq. 12a) contains the second answer (Eq. 12b), but the first answer is an uncertain answer that has infinite possible solutions in the interval [0.25,4]. The second answer is absolutely crisp and is not a fuzzy number, which indicates that a fuzzy number divided by a fuzzy number is not necessarily a fuzzy number. The second case was not addressed in Kaufmann and Gupta[9] and Moore.[10] Therefore, the definitions of fuzzy subtraction and fuzzy division need to be revised to produce crisp results in such cases.

The modified definition of the fuzzy division for A_α/A_α is hereby given for $A_\alpha = [a,b]$ with $0 \notin [a,b]$. For all x, $y \; E \; A_\alpha$, the following definition can be made:

1. For the condition that x and y are arbitrary, the original definition of fuzzy division as provided in Kaufmann and Gupta[9] and Moore[10] can be used as follows:

$$\frac{A_\alpha(x)}{A_\alpha(y)} = \frac{[a, b]}{[a, b]}$$
$$= [\min(a/a, a/b, b/a, b/b), \max(a/a, a/b, b/a, b/b)] \quad (13a)$$

2. For the case $x = y$, the fuzzy division should be defined as follows:

$$\frac{A_\alpha(x)}{A_\alpha(y)} = \frac{[a, b]}{[a, b]} = 1 \quad (13b)$$

For fuzzy subtraction, a similar definition for $A_\alpha - A_\alpha$ can be given for all x, $y \; E \; A_\alpha$ as follows:

1. For the case that x and y are arbitrary, the fuzzy subtraction is

$$A_\alpha(x) - A_\alpha(y) = [a,b] - [a,b] = [a - b, b - a] \quad (14a)$$

2. For $x = y$, the fuzzy subtraction is

$$A_\alpha(x) - A_\alpha(y) = [a,b] - [a,b] = 0 \quad (14b)$$

To illustrate the use of fuzzy arithmetic, the following simultaneous equations with fuzzy coefficients are solved using the Gaussian elimination method:

$$[4,6]x + [1,3]y = 9 \quad (15a)$$
$$[1,3]x + [4,6]y = 12 \quad (15b)$$

The augmented matrix is given by

$$\left[\begin{array}{cc|c} [4, 6] & [1, 3] & [9, 9] \\ [1, 3] & [4, 6] & [12, 12] \end{array}\right] \quad (16a)$$

Each term in the first row is divided by [4, 6]. By the modified definition of fuzzy division in Eq. (13b), the value of the first term in the first row becomes 1, that is, [4, 6]/[4, 6] = 1. Thus the augmented matrix becomes

$$\left[\begin{array}{cc|c} 1 & [1/6, 3/4] & [9/6, 9/4] \\ [1, 3] & [4, 6] & [12, 12] \end{array}\right] \quad (16b)$$

The first row is multiplied by [1,3] and is subtracted from the second row for each column. Similarly, using Eq. (14b), the value of the first term in the second row becomes 0, that is, [1,3] − 1 × [1,3] = 0, resulting in

$$\begin{bmatrix} 1 & [1/6,\ 3/4] & [3/2,\ 9/4] \\ 0 & [7/4,\ 35/6] & [21/4,\ 21/2] \end{bmatrix} \quad (16c)$$

Now, y can be determined from Eq. (16c) as follows:

$$[7/4,\ 35/6]y = [21/4,\ 21/2] \quad (16d)$$

Therefore, y is given by

$$y = \frac{[21/4,\ 21/2]}{[7/4,\ 35/6]} = [9/10,\ 6] = [0.9,\ 6] \quad (16e)$$

Similarly, x can be determined from Eq. (16c) as follows:

$$x + [1/6,3/4]y = [3/2,9/4] \quad (16f)$$

Solving for x produces

$$\begin{aligned} x &= [3/2,9/4] - [1/6,3/4] \times [9/10,6] \\ &= [3/2,9/4] - [3/20,9/2] \\ &= [-3,42/20] = [-3,2.1] \end{aligned} \quad (16g)$$

An alternate solution procedure can be used without dividing the first row in Eq. (16a) by [4,6]. The range of y in this case is not changed, that is, y = [0.9,6], whereas the backward substitution for x is given by

$$[4,6]x + [1,3]y = [9,9] \quad (16h)$$

Solving for x produces

$$\begin{aligned} x &= \frac{1}{[4,\ 6]}([9,\ 9] - [1,\ 3] \times [9/10,\ 6]) \\ &= \frac{1}{[4,\ 6]}([9,\ 9] - [9/10,\ 18]) \\ &= \frac{[-9,\ 81/10]}{[4,\ 6]} = [-9/4,\ 81/40] = [-2.25,\ 2.025] \end{aligned} \quad (16i)$$

From the results in Eqs. (16g) and (16i) it can be seen that different computation procedures using fuzzy arithmetic can generate different solutions. However, the exact solution of Eqs. (15a) and (15b) is y = [1,3] and x = [0,2], which can be obtained by the method based on permutations, as described in the next section. The result of fuzzy arithmetic generally has a wider interval that contains the exact range.[10]

In summary, structural analysis based on fuzzy arithmetic can be performed by properly replacing the arithmetic of structural analysis with the corresponding fuzzy arithmetic. However, the proposed definitions of fuzzy division and fuzzy subtraction need to be used.

4.2 Analysis based on permutations

The analysis based on permutations can be developed using Eq. (8), $[K]\{d\} = \{F\}$, which is a set of *linear* simultaneous equations. In Eq. (8), each nonzero element contains E and is a linear function of E. To get the extreme values for each component in $\{d\}$, one can substitute the maximum and minimum values of E for each structural member into Eq. (8) to obtain the different permutations of the coefficients for $[K]$ and solve Eq. (8) by the Gaussian elimination method. If a structure has m members, the worst case is that E in every member varies differently from the rest of members, and the Gaussian elimination method must be applied 2^m times. The algorithm can be summarized as follows: (1) generate the permutations for E using extreme values, (2) substitute the E values in the permutations into $[K]$ and solve Eq. (8) for $\{d\}$ and member forces using the Gaussian elimination method, and (3) find maximum and minimum values for each displacement and member force among all these possible cases obtained through permutations.

For a set of n linear simultaneous equations with fuzzy coefficients, it generally requires $2^{n\times n}$ permutations to generate all possible extreme solutions, provided that the coefficients are independent. For example, Eq. (15) has 2 unknowns and needs $2^{2\times 2} = 16$ permutations to find the maximum and minimum values for x and y. Figure 2 shows the geometric solutions of Eq. (15) by the permutations method. Using the extreme values of the fuzzy coefficients in Eq. (15a) produces four linear equations $4x + y = 9$, $4x + 3y = 9$, $6x + y = 9$, and $6x + 3y = 9$, as shown in Fig. 2 in dotted lines. Similarly, the four solid lines $x + 4y = 12$, $x + 6y = 12$, $3x + 4y = 12$, and $3x + 6y = 12$ as shown in Fig. 2 are obtained from Eq. (15b). In this case, each intersection of the four solid lines and the four dotted lines is a solution of the permutations using the extreme values of the fuzzy coefficients. It can be shown that the range of x is [0,2] and the range of y is [1,3]. The proposed permutations method for solving linear simultaneous equations is similar in concept to the vertex method[6] for evaluating functions of fuzzy variables.

5 EXAMPLES

Four examples are used in this section to illustrate the use of the proposed methods. The four typical structures are (1) a determinate truss, (2) an indeterminate truss, (3) a beam, and (4) a frame. In these examples, a normalized E, i.e., $E = 1$ when $\alpha = 1$, is used. Representative values can be obtained by substituting the value of E into the related equations. In the determinate truss example, the results of both the fuzzy arithmetic and permutations methods are shown, whereas for the other examples, only the results using the permutations method are shown.

Example 1: Determinate truss

A determinate truss and its loading condition are shown in Fig. 3a.[7] There are three members as indicated. Assume that $L/A = 1$ for all members, where L = length and A =

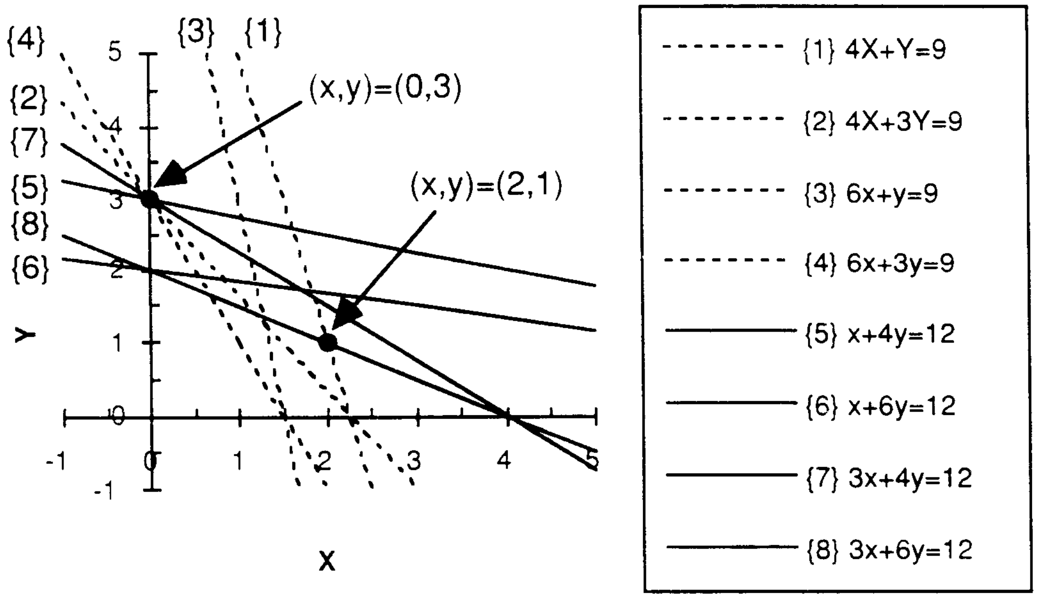
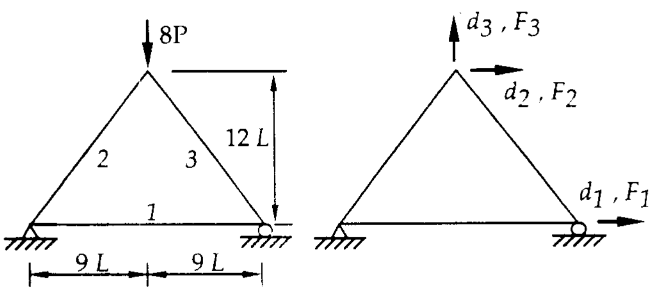


Fig. 2. Geometric solutions of $[4,6]x + [1,3]y = 9$ and $[1,3]x + [4,6]y = 12$.



(a) Determinate Truss (b) Degrees of Freedom

Fig. 3. Example determinate truss and its degrees of freedom.

cross-sectional area. The fuzzy E value for each member is a triangular fuzzy number with the triplet [0.9,1.0,1.1]. Figure 3b shows the possible positive nodal displacements d_1, d_2, and d_3 and the corresponding positive nodal forces F_1, F_2, and F_3. The nodal force vector is

$$\{F\} = \{F_1, F_2, F_3\}^T = P\{0,0,-8\}^T \quad (17a)$$

The results with $E_{\alpha=1} = 1$ for the displacement vector $\{d\}$ and the member force vector $\{Q\}$ are, respectively, given by

$$\{d\} = \{d_1, d_2, d_3\}^T = (PL/AE)\{3,1.5,-7.375\}^T \quad (17b)$$

and

$$\{Q\} = \{Q_1, Q_2, Q_3\}^T = P\{3,-5,-5\}^T \quad (17c)$$

A positive member force indicates a tensile force, and a negative member force indicates a compressive force. The results of the displacements and the member forces for both methods are shown in Figs. 4 and 5, respectively.

The permutations method shows that the displacements are almost triangular fuzzy numbers. The member forces according to this method have crisp values.

The fuzzy arithmetic method produces the displacements and the member forces that are both fuzzy numbers, as shown in Figs. 4 and 5, respectively. As mentioned before, the fuzzy arithmetic method usually produces wider intervals,[10] i.e., larger upper bounds and smaller lower bounds, which contain the exact range according to permutations method.

Example 2: Indeterminate truss

An indeterminate truss to the first degree of redundancy and its loading condition are shown in Fig. 6a.[7] Assume again that $L/A = 1$ for all members. The fuzzy E value is [0.8,1.0,1.2] for each member. Figure 6b shows the possible positive nodal displacements d_1, d_2, d_3, and d_4 and the corresponding positive nodal forces F_1, F_2, F_3, and F_4. The nodal force vector is

$$\{F\} = \{F_1, F_2, F_3, F_4\}^T = P\{10,10,10,0\}^T \quad (18a)$$

The results with $E_{\alpha=1} = 1$ for the displacement vector $\{d\}$ and the member force vector $\{Q\}$ are, respectively, given by

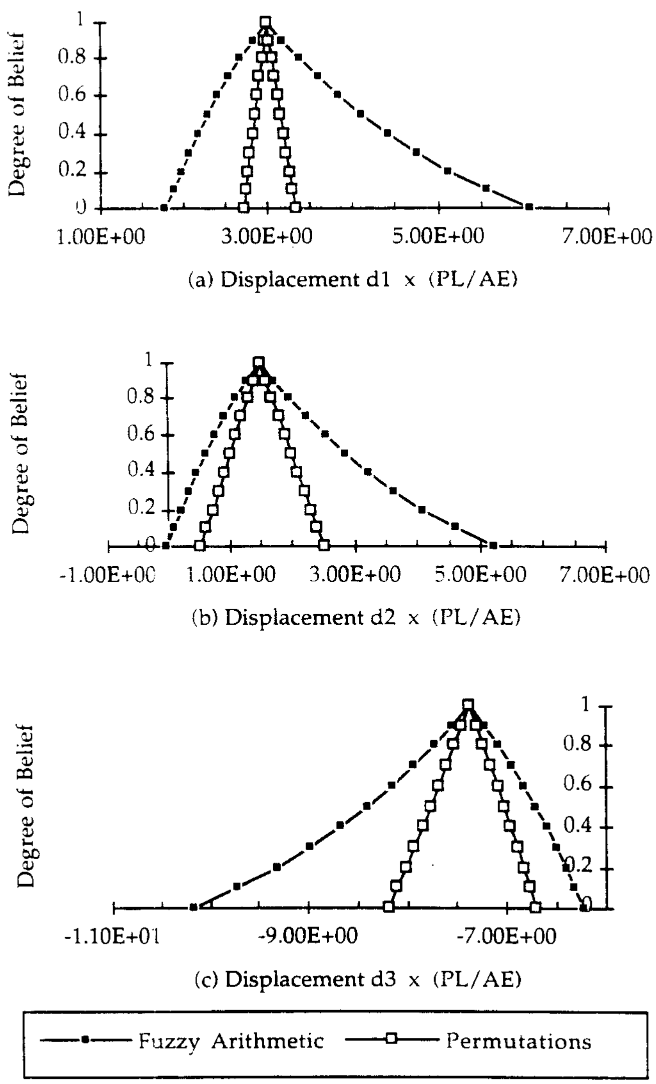


Fig. 4. Results of determinate truss—displacements.

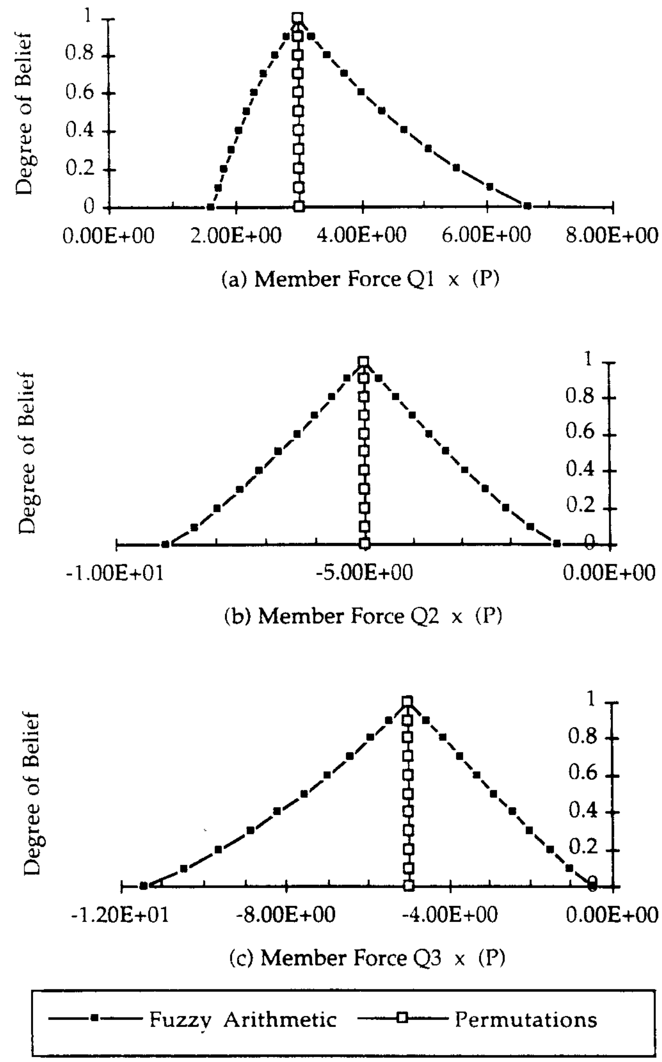


Fig. 5. Results of determinate truss—member forces.

$$\{d\} = \{d_1, d_2, d_3, d_4\}^T$$
$$= (PL/AE)$$
$$\{25.714, 16.429, 24.286, -8.5714\}^T \quad (18b)$$

and

$$\{Q\} = \{Q_1, Q_2, Q_3, Q_4, Q_5\}^T$$
$$= P\{16.429, -1.4286, -8.5714,$$
$$14.286, -10.714\}^T \quad (18c)$$

The results of the displacements and the member forces are shown in Fig. 7a and b, respectively.

In this case, the resulting displacements and member

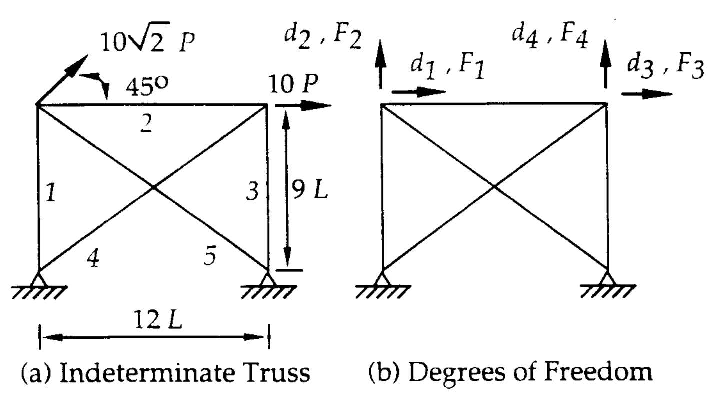


Fig. 6. Example indeterminate truss.

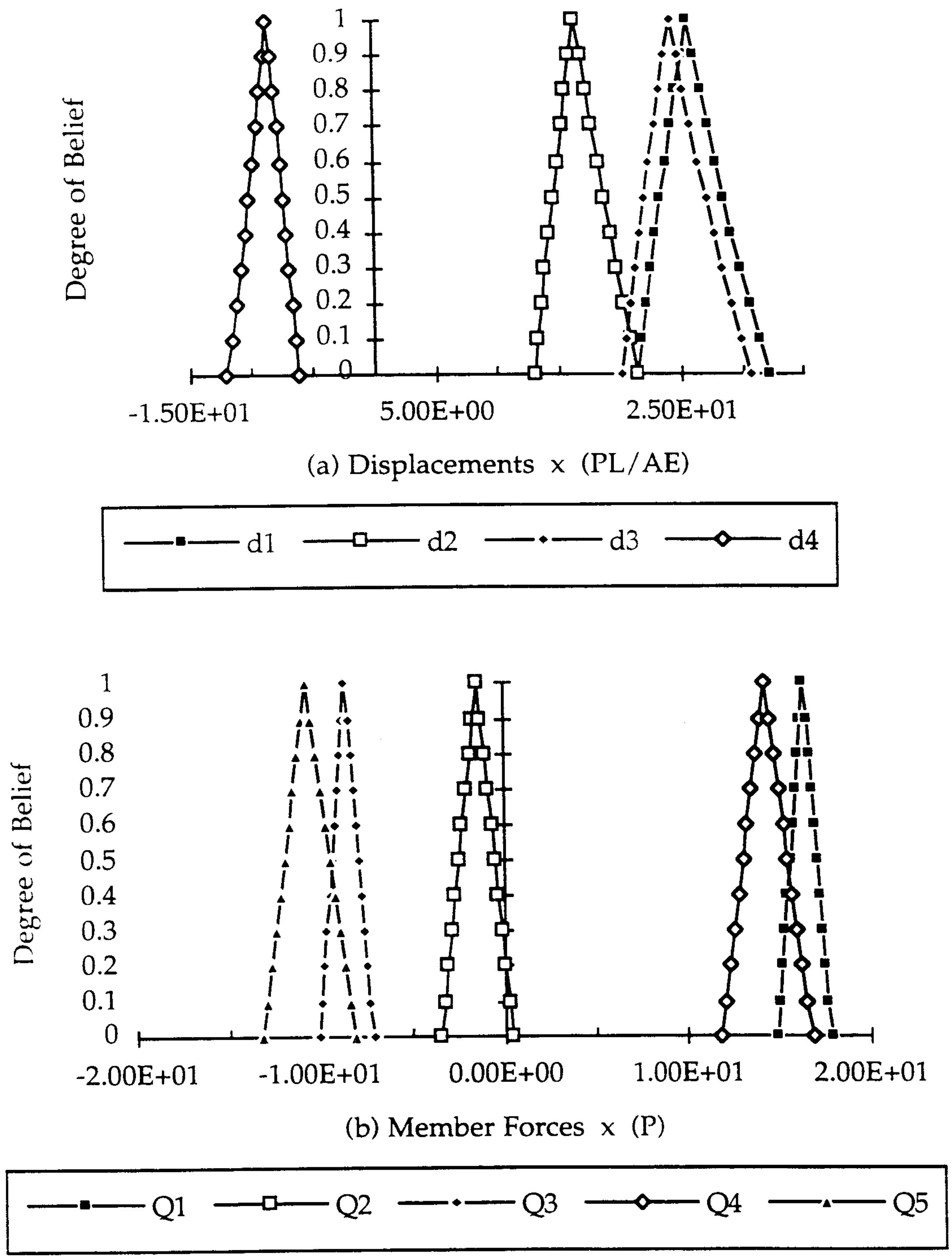


Fig. 7. Results of indeterminate truss.

forces are fuzzy. The member forces are triangular fuzzy numbers. Also, when α approaches 0, the member force Q_2 may change from compression to tension.

Figure 8 shows the normalized member forces, that is, $Q_\alpha/Q_{\alpha=1}$. The figure shows that a fuzzy E = [0.9,1.0,1.1] produces triangular fuzzy member forces with the normalized values Q_1 = [0.91,1.00,1.09], Q_2 = [−0.35,1.00,2.42], Q_3 = [0.83,1.00,1.18], Q_4 = [0.83,1.00,1.18], and Q_5 = [0.76,1.00,1.22]. In this case, some member forces contain more uncertainty than other member forces.

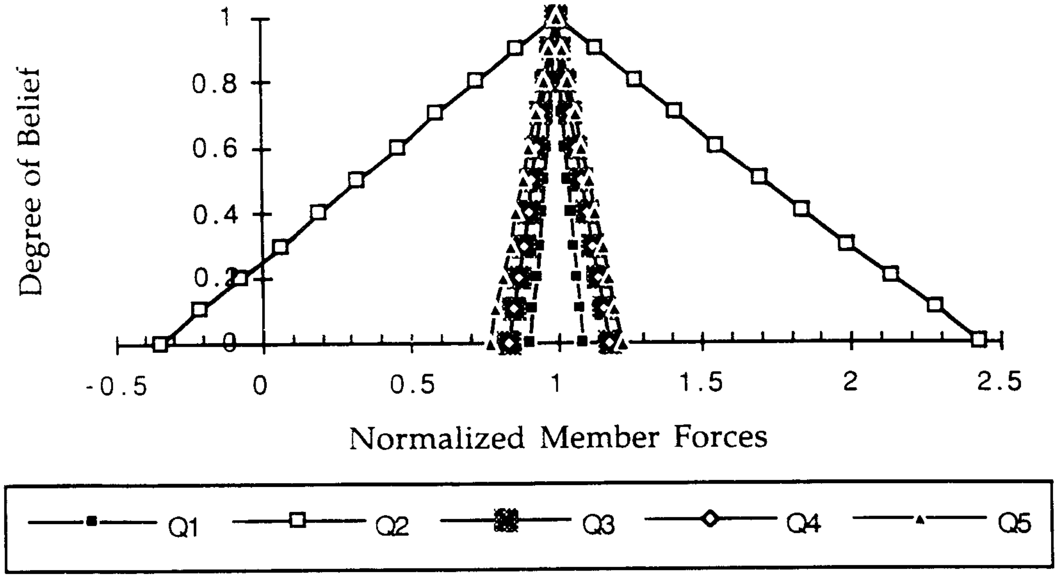


Fig. 8. Normalized member forces for indeterminate truss.

Example 3: Continuous beam

A two-span continuous beam and its loading condition are shown in Fig. 9a.[12] Assume that $I/L = 1$ for all members, where I = moment of inertia. The fuzzy E value is [0.8,1.0,1.2] for each member. Figure 9b shows the following information: (1) the possible nodal displacements (rotations) d_1 and d_2, (2) the corresponding nodal forces F_1 and F_2, (3) the member forces (moments) Q_1, Q_2, Q_3, and Q_4, and (4) the support reactions R_1, R_2, and R_3. The positive directions are as indicated in Fig. 9b. The nodal force vector is

$$\{F\} = \{F_1,F_2\}^T = PL\{-1.125,-0.125\}^T \quad (19a)$$

The results with $E_{\alpha=1} = 1$ for the displacement vector $\{d\}$, the moments vector $\{Q\}$, and reaction vector $\{R\}$ are, respectively, as follows:

$$\{d\} = \{d_1, d_2\}^T = (PL^2/112EI)\{-5, 117\}^T \quad (19b)$$

$$\{Q\} = \{Q_1, Q_2, Q_3, Q_4\}^T$$
$$= (PL/112)\{-62, -40, -72, 0\}^T \quad (19c)$$

$$\{R\} = \{R_1, R_2, R_3\}^T = (P/56)\{107, 69, -64\}^T \quad (19d)$$

The results of the displacements (rotations) and the member forces (moments and reactions) are shown in Fig. 10a, b, and c, respectively. The resulting displacements are fuzzy numbers, as shown in Fig. 10a, and both the moments and the reactions are almost triangular fuzzy numbers, as shown in Fig. 10b and c, respectively.

It can be noted from these figures that the permutations

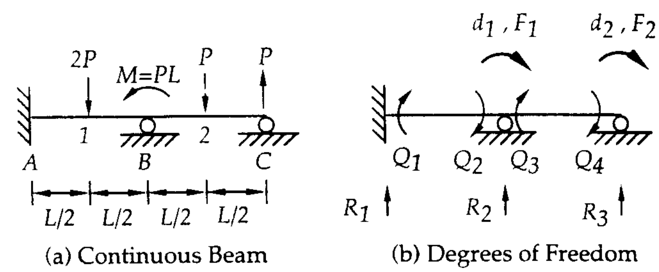


Fig. 9. Example continuous beam and its degrees of freedom.

method produces a result that satisfies the boundary condition at support C, as shown in Fig. 9a; i.e., the moment $Q_4 = 0$ according to Fig. 10b. This boundary condition at support C is not satisfied according to the fuzzy arithmetic method by producing a fuzzy number for Q_4.

For E = [0.8,1.0,1.2], the normalized moments are Q_1 = [0.90,1.00,1.09], Q_2 = [0.70,1.00,1.28], and Q_3 = [0.84,1.00,1.17], as shown in Fig. 11a, and the normalized reactions are R_1 = [0.92,1.00,1.08], R_2 = [0.79,1.00,1.22], and R_3 = [0.91,1.00,1.09], as shown in Fig. 11b. In this case, the normalized member forces show about the same uncertainty level as E.

Example 4: Frame

A three-member frame is shown in Fig. 12a.[7] Each member length is $10L$, and member 2 is subjected to a downward uniform load with intensity $12w$. The fuzzy E value is [0.8,1.0,1.2] for each member. Figure 12b shows (1) the

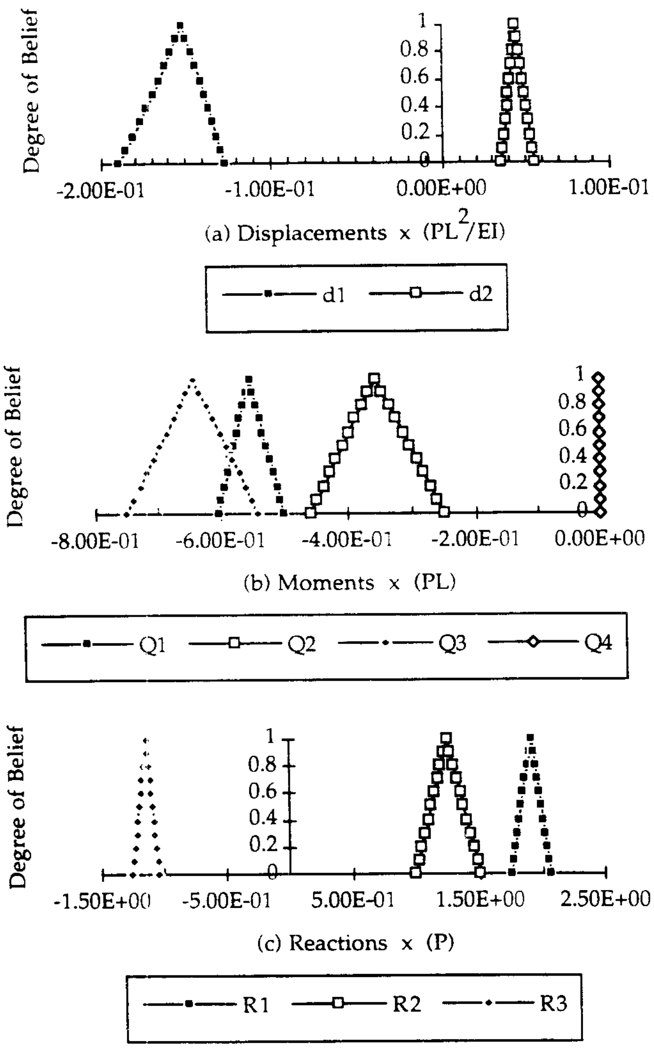


Fig. 10. Results of continuous beam.

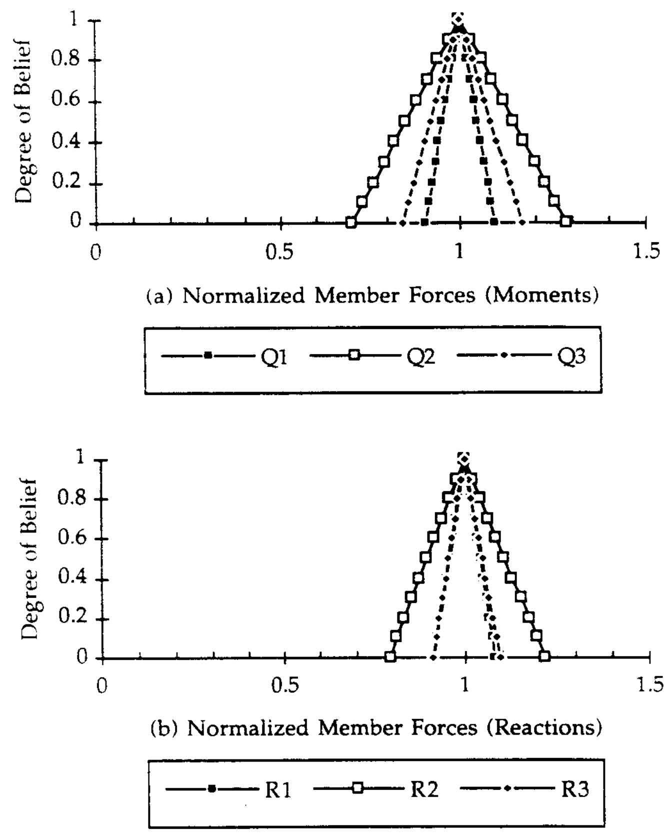


Fig. 11. Normalized member forces for continuous beam.

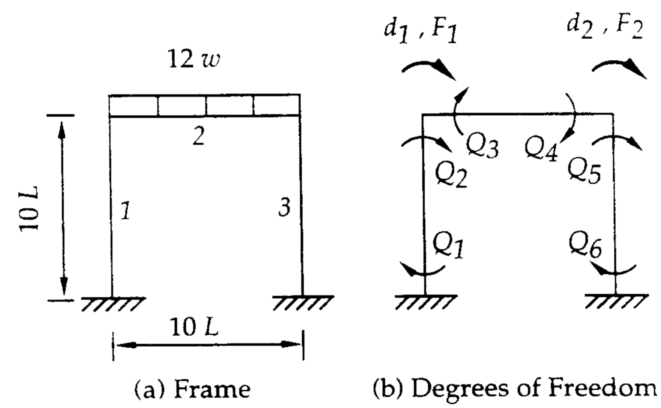


Fig. 12. Example frame and its degrees of freedom.

possible positive nodal displacements d_1 and d_2 and the corresponding positive nodal forces F_1 and F_2 and (2) the positive member forces Q_1, Q_2, Q_3, Q_4, Q_5, and Q_6. The nodal force vector is

$$\{F\} = \{F_1, F_2\}^T = (wL^2)\{100, -100\}^T \quad (20a)$$

The results with $E_{\alpha=1} = 1$ for the displacement vector $\{d\}$ and the member force vector $\{Q\}$ are, respectively, given by

$$\{d\} = \{d_1, d_2\}^T = (wL^3/3EI)\{50, -50\}^T \quad (20b)$$

$$\{Q\} = \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}^T$$
$$= (wL^2)\{33.3, 66.7, -66.7, 66.7, -66.7, -33.3\}^T \quad (20c)$$

The results of the displacements (rotations) and the member forces (moments) are shown in Fig. 13a and b, respectively. Again, the results show that the displacements are fuzzy numbers, as shown in Fig. 13a, and the member forces (moments) are almost triangular fuzzy numbers, as shown in Fig. 13b.

Figure 14 shows the normalized member forces. In this particular case with symmetrical structure and symmetrical loading, the normalized member forces are the same, that is, Q = [0.81,1.00,1.18], which is almost the same as E.

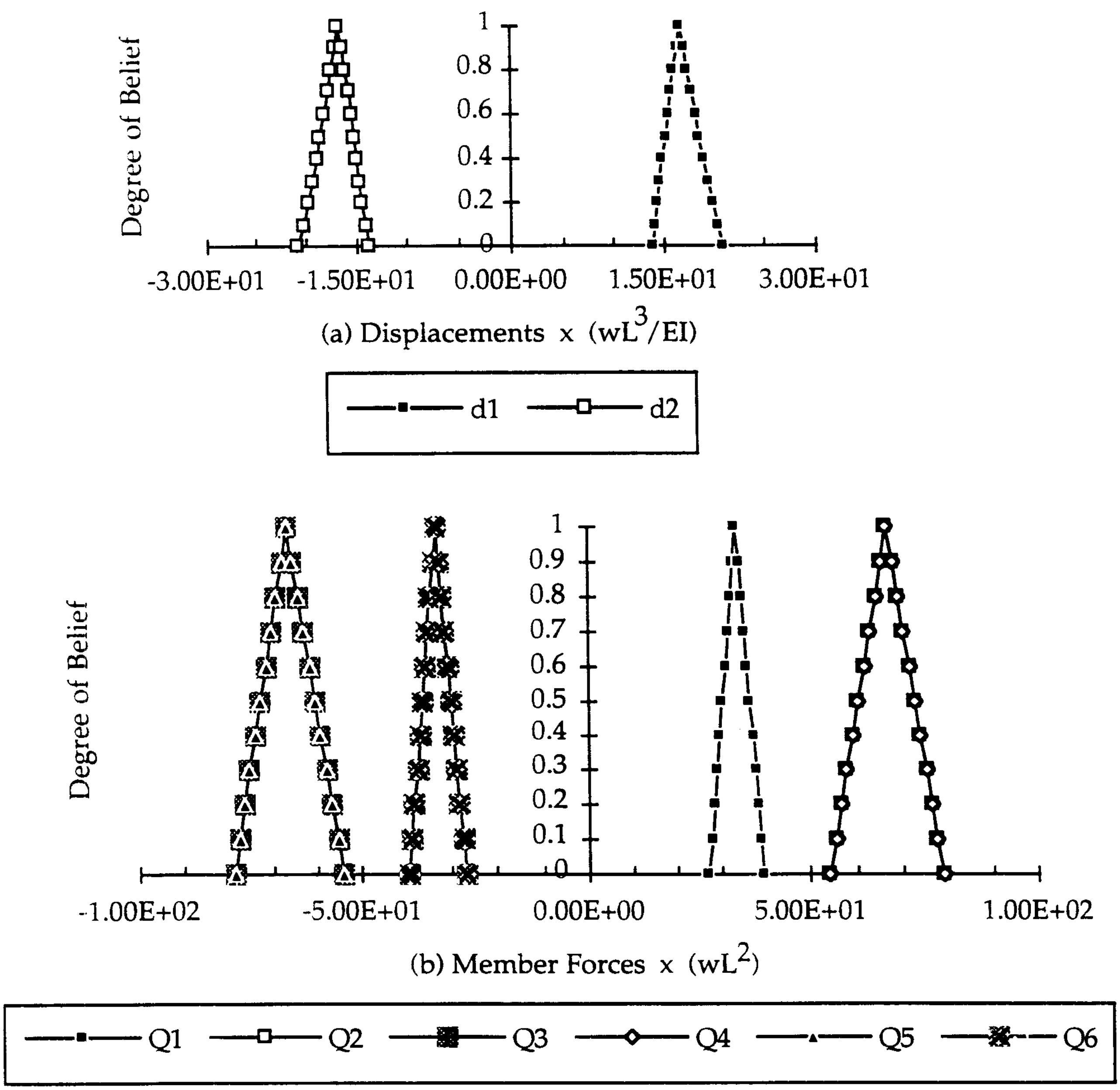


Fig. 13. Results of frame.

6 SUMMARY AND CONCLUSIONS

Structural analysis methods are reexamined in this paper by considering the cognitive type of uncertainty. Two analytical methods are developed in this paper, the fuzzy arithmetic method and the permutations method. The fuzzy arithmetic method requires evaluation of the Gaussian elimination method only once for each alpha-cut but gets approximate solutions that have generally wider intervals containing the exact ranges. The permutations method uses all extreme values of any uncertain variables to get the exact solution but requires more computing time. For the purpose of illustration, the modulus of elasticity E is modeled as a triangular fuzzy number, and the structural behavior due to this uncertainty was investigated.

The following observations can be made based on the examples using the permutations method with a triangular fuzzy number for E:

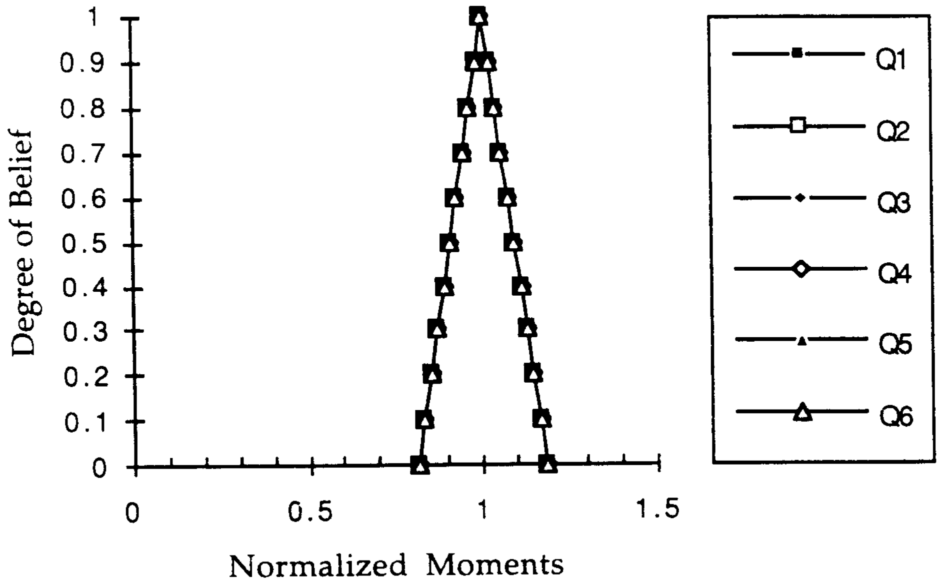


Fig. 14. Normalized member forces for frame.

1. For determinate trusses, the member forces are still crisp values. For indeterminate trusses, the member forces become fuzzy numbers.
2. For all types of structures, the member forces can be either crisp values or triangular fuzzy numbers. The displacements are always fuzzy numbers and are not triangular fuzzy numbers.
3. In indeterminate structures, increasing the uncertainty in E may result in the reversal of structural behavior (i.e., sign change) for some members.

Future studies are needed in this area to improve the computational efficiency of the permutations method. Also, the combined effects of both types of uncertainty, i.e., cognitive and noncognitive, on structural behavior need to be explored.

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