Moment-Rotation Characteristic of Slabs

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ABSTRACT

This paper presents a method to find the moment-rotation characteristic of a reinforced concrete slab or beam. The method is based on Newton-Raphson iteration of the forces that are represented in finite-difference form. The non-linear behavior of steel and concrete are taken into consideration in the proposed method. The method is illustrated by computing the moment-rotation curve of a typical highway bridge slab, and the result compared to the results of a finite element analysis. The results show that the approximate method proposed generates results that closely match the result of a non-linear finite element analysis.

KEYWORDS

Moment-Curvature, Ductility, Reinforced Concrete, Welded Steel Mesh

I - Introduction

Load-deflection behavior of reinforced concrete members is often characterized by their moment-rotation relationship. Consideration of the moment-rotation characteristic is important in design for the following reasons:

1. The moment-rotation curve can be used as a measure of the ductility of a member;
2. It shows the geometric instabilities of a member, if they exist; and
3. It shows whether the concrete crushes before or after yielding of the reinforcement steel.

If the concrete cracks in tension before the reinforcement steel yields and if the ultimate flexural capacity of the beam is limited by the tensile strength of the steel, then the moment-rotation relationship can be assumed to be trilinear [Park and Paulay, 1975]. The moments and curvatures at first cracking of concrete, at first yielding of steel, and at the ultimate capacity of the beam can be calculated [Gurfinkel and Robinson, 1967], [Pfrang et al., 1964]. These calculations are usually based on the assumption that the steel is elasto-perfectly-plastic in order to simplify the calculation. Curvature of the member at ultimate capacity is usually not available or erroneous because of this assumption. A more realistic representation of the steel's behavior, however, requires a more computationally intensive procedure that should be automated. An example of the importance of the moment-rotation relationship can be illustrated by the use of welded steel mesh as structural reinforcement. Because the cold-drawn steel used for the mesh does not have a well defined yield point and is less ductile than the mild steel usually used for concrete reinforcement, an accurate moment-rotation relationship is needed to predict whether or not the welded steel mesh reinforced member has the desirable characteristics.

This paper develops an approximate method that finds the moment-rotation relationship through the use of finite-difference approximation and Newton-Raphson iteration technique. The results are then compared to the results of finite element analysis.

II - Mathematical Formulation

Assuming that plane sections remain plane after deformation and the strain distribution is linear, then, for a given concrete strain in the extreme compression fiber ϵ_c and neutral axis depth a, the strain of the steel ϵ_i can be determined (see Fig. 1):

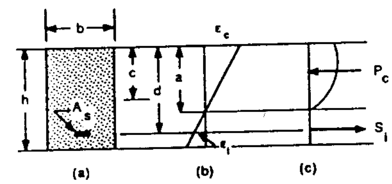

Figure 1 - (a) Cross Section, (b) Strain Distribution, and (c) Stress Distribution.

$$\epsilon_i = \epsilon_c \frac{a - d_i}{a} \tag{1}$$

in which d_i is the depth of steel layer i from the top extreme fibers. The stress σ_i corresponding to ϵ_i may be found from the stress-strain curve of the steel. Then the steel force S_i may be found by:

$$S_i = \sigma_i A_{s_i} \tag{2}$$

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in which A_{s_i} is the area of the steel at layer i.

The distribution of the concrete stress over the cross section and the resultant force may be found from the stress-strain diagram of the concrete and steel. Assuming that the stress-strain relationship of the concrete is known, then for given values of concrete strain, ϵ_c, and the location of the neutral axis, a, the concrete compression force P_c, for a rectangular cross section is given by the following equation:

$$P_c = \frac{a}{\epsilon_c} \int_o^{\epsilon_c} b\sigma_c(\epsilon)d\epsilon - \frac{a}{\epsilon_c} \int_o^{\epsilon_x} bE_c\epsilon d\epsilon \quad (3)$$

and the corresponding moment M_c is given by:

$$M_c = (\frac{a}{\epsilon_c})^2 \int_0^{\epsilon_c} b\sigma_c(\epsilon)d\epsilon + \int_0^{\epsilon_x} bE_c\epsilon^2 d\epsilon \quad (4)$$

in which ϵ_x is limited to the value of strain ϵ_T when the concrete fails in tension, i.e.:

$$\begin{aligned} &\text{if} \quad \epsilon_x < \epsilon_T; \quad \text{then} \quad \epsilon_x = \epsilon \\ &\text{if} \quad \epsilon_x \geq \epsilon_T; \quad \text{then} \quad \epsilon_x = 0 \end{aligned} \quad (5)$$

In Eqs. (3) and (4), the stress-strain relationship of the concrete in tension is assumed to be linear. The steel is assumed to have an elasto-plastic relationship with elastic modulus, E_s, and plastic modulus, E_p; it is also assumed that the steel reaches an ultimate strain of ϵ_u. Therefore, given the yield point (σ_y, ϵ_y) and the ultimate point (σ_u, ϵ_u), the stress-strain relationship of the steel is completely defined. If the actual stress-strain relationship of the steel is known, it can be used here instead of the elasto-plastic relationship. By adding the forces due to the steel reinforcement to the concrete forces, the total axial force and moment of a cross section is obtained as shown below:

$$P = P_c + \sum_{i=1}^{N} \sigma_i A_{s_i} \quad (6)$$

$$M = M_c + \sum_{i=1}^{N} \sigma_i A_{s_i}(c - d_i) - P(a - c) \quad (7)$$

in which c is the distance from the top extreme fiber to the center of gravity of the cross section. Therefore, by specifying a set of values of concrete strain in the extreme compressive fiber ϵ_c and the curvature ϕ, one can calculate the axial force and moment of the cross section via Eqs. (6) and (7), where the curvature, ϕ, is equal to the concrete strain, ϵ_c, divided by the centroidal distance, a. Let us denote this point by $(\bar{P}, \bar{M})$. The equilibrium point $(\bar{P}, \bar{M})$ obtained is not likely to have the correct axial force i.e., $\bar{P}$ is not equal to the axial load on the actual cross section. The axial load and moment will be corrected to a specified value P^* and M^* by modifying the strain and curvature of the cross section.

The Newton-Raphson technique is used in which the force and moemnt are expressed in terms of the Taylor series with the terms containing the second and higher derivatives omitted. Then P and M are given by:

$$P = \bar{P}(\epsilon, \phi) + \frac{\partial P(\epsilon, \phi)}{\partial \phi}\delta\phi + \frac{\partial P(\epsilon, \phi)}{\partial \epsilon}\delta\epsilon \quad (8)$$

$$M = \bar{M}(\epsilon, \phi) + \frac{\partial M(\epsilon, \phi)}{\partial \phi}\delta\phi + \frac{\partial M(\epsilon, \phi)}{\partial \epsilon}\delta\epsilon \quad (9)$$

The step size used are chosen as $\Delta\phi = \alpha\phi + \beta$ and $\Delta\epsilon = \alpha\epsilon + \beta$, in which β is a small number and α is the percentage of increment. Using these step sizes, the first-order finite difference representation of the partial derivatives are:

$$\begin{aligned} \frac{\partial P}{\partial \phi} &= \frac{P(\phi + \Delta\phi, \epsilon) - \bar{P}}{\Delta\phi} \\ \frac{\partial P}{\partial \epsilon} &= \frac{P(\phi, \epsilon + \Delta\epsilon) - \bar{P}}{\Delta\epsilon} \\ \frac{\partial M}{\partial \phi} &= \frac{M(\phi + \Delta\phi, \epsilon) - \bar{M}}{\Delta\phi} \\ \frac{\partial M}{\partial \epsilon} &= \frac{M(\phi, \epsilon + \Delta\epsilon) - \bar{M}}{\Delta\epsilon} \end{aligned} \quad (10)$$

$\delta\phi$ and $\delta\epsilon$ can be solved by substituting Eq. (10) into Eqs. (8) and (9) and setting the axial load and moment equal to the specified values (P^*, M^*). Then the new approximation to ϵ and ϕ becomes:

$$\begin{aligned} \epsilon_{new} &= \epsilon + \delta\epsilon \\ \phi_{new} &= \phi + \delta\phi \end{aligned} \quad (11)$$

To check for convergence, the new values of ϵ and ϕ obtained in Eq. (11) are substituted into Eqs. (6) and (7), to obtain a new set of values of axial load and moment (P_{new}, M_{new}). If (P_{new}, M_{new}) is sufficiently close to the specified value (P^*, M^*), then the desired equilibrium point with the corresponding ϵ and ϕ has been obtained. If the new point (P_{new}, M_{new}) is not sufficiently close to the specified value, an iterative process with ϵ_{new} and ϕ_{new} being the new initial point is initiated until convergence is acheieved. A listing of the program to calculate the section curvature, ϕ, corresponding to a specified axial force, P^*, and bending moment, M^*, is listed in the Appendix C. Sample input and output data are also listed in Appendices D and E, respectively.

III - Numerical Example

The example used is a typical section of a bridge slab used by the Maryland Department of Transportation. The slab is eight inches thick with one inch cover on the bottom. To understand the flexural behavior of the slab, a typical section of the slab with unit width is analyzed. Assuming that the slab is simply supported as shown in Fig. 2, then the axial force, P, is equal to zero. The cross section of the slab with unit-width and the stress-strain characteristics of steel and concrete are shown in Fig. 3.

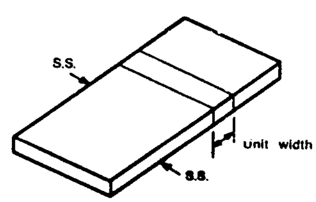

Figure 2 - Typical Slab Used in Bridge Deck.

To obtain the moment-curvature characteristics of this slab, the axial force P is set to zero and a value of the moment not equal to zero is specified. A set of initial values for strain and rotation, ϵ_c and ϕ, is assumed. When convergence is achieved, the final values of curvature and concrete strain are obtained, and the location of the neutral axis of the section is calculated by:

$$a = \frac{\epsilon_c}{\phi} \quad (12)$$

The concrete strain in the extreme compression fiber, ϵ_c, and the section curvature, ϕ, are obtained for a set of specified values of M and P.

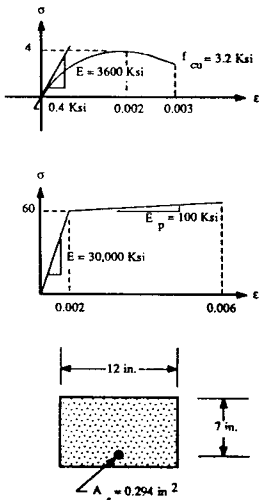

Figure 3 - Stress Strain Characteristics of Concrete and Steel

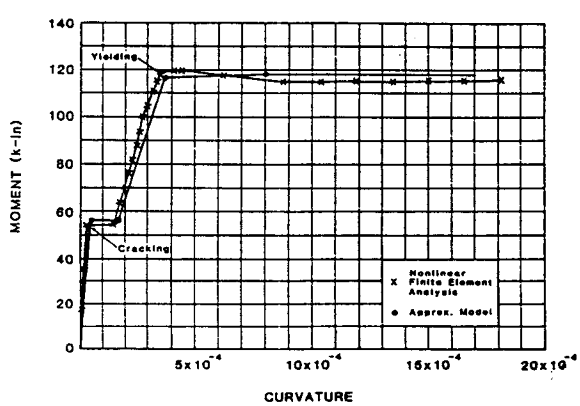

Figure 4 - Moment Rotation Characteristic of Slab.

Further increase in the moment did not result in convergence, implying that the steel has reached the ultimate strain. The resulting moment-rotation characteristic is shown in Fig. 4. The moment-rotation relationship is then calculated using a finite element program. Two-dimensional elements with nonlinear concrete material model are used. The results are compared to the results of the proposed algorithm in Fig. 4. The results of the proposed method compare well with the results from the nonlinear finite element analysis.

IV - Conclusions

Based on the results of the numerical example, the following conclusions are made:

1. The ultimate moment and moment at first yield calculated using the procedure proposed herein are approximately equal to those values calculated by a more rigorous finite element analysis.
2. The ultimate rotation can be obtained by the program developed (Appendix C).
3. The cracking moment and the corresponding rotation can also be obtained by the program.
4. The moment-curvature characteristic for a beam with a non-linear stress-strain relationship for the concrete and steel can be readily obtained.

Appendix A - References

[1] Gurfinkel G. and Robinson A., "Determination of Strian Distribution and Curvature in a Reinforced Concrete Section Subjected to Bending Moment and Longitudinal Load," ACI Journal, July, 1967, pp. 398-403.

[2] Park R. and Paulay T., **Reinforced Concrete Structures**, John Wiley & Sons, New York, NY., 1975, pp. 196-202.

[3] Pfrang E.O., Siess C.P. and Sozen, M.A., "Load-Moment-Curvature Characteristics of Reinforced Concrete Cross Sections," ACI Journal, Vol. 61, No. 7, July 1964, pp. 763-776.

Appendix B - Notation

a = Distance from extreme compressive fiber to neutral axis
A_{s_i} = Area of stee at layer i
c = Distance from extreme compressive fiber to centroid
d_i = Depth of steel layer i
M_c = Bending moment resisted by concrete
P_c = Compressive force resisted by concrete
S_i = Force of steel at layer i
ϵ_c = Strain of concrete at the extreme compressive fiber
ϵ_i = Strain of steel at layer i
ϵ_T = Maximum strain of concrete in tension
ϵ_x = Strain of concrete
ϕ = Rotation of the cross section
σ_i = Stress of steel at layer i

Appendix C - Program Listing

```
C------------------------------------------------------------
C                    MOMENT-CURVATURE RELATIONSHIP
C     THIS PROGRAM DETERMINES THE STRAIN DISTRIBUTION AND THE
C   CURVATURE IN REINFORCED CONCRETE SECTION SUBJECTED TO
C   BENDING MOMENT AND LONGITUDINAL LOAD.
C     THIS PROGRAM CONSIDERS TENSION IN THE CONCRETE.
C------------------------------------------------------------
      DIMENSION EE(100).FF(100)
      REAL EE.FF.P.M.H.B.D.DC.AS.ASC.FY.EY
      REAL FC.E0.E4.EC.ER.FR.EX
      REAL FI.E4I.ALPHA.BETA
      REAL E1.E2.E3.FS1.FS2.FS3
      REAL E1I.E2I.E3I.FS1I.FS2I.FS3I
      REAL E1IF.E2IF.E3IF.FS1IF.FS2IF.FS3IF
      REAL E1IE.E2IE.E3IE.FS1IE.FS2IE.FS3IE
      REAL E1NEW.E2NEW.E3NEW.FS1NEW.FS2NEW.FS3NEW
      REAL DFI.DE4I
      REAL PI.MI.PIF.MIF.PIE.MIE
      REAL PNEW.MNEW.FNEW.E4NEW
      OPEN(20.FILE-'MOMCUR.IN'.STATUS-'OLD')
      OPEN(21.FILE-'MOMCUR.OUT'.STATUS-'NEW')
       NCYCLE-0
      DO 10 J-1.100
       READ(20.*) EE(J).FF(J)
       IF(FF(J).EQ.9999) GO TO 20
10    CONTINUE
20    READ(20.*) P.M
      READ(20.*) H.B.C.D.DC
      READ(20.*) AS.ASC.FY.EY
      READ(20.*) FC.E0.EU
      READ(20.*) FI.E4I.ALPHA.BETA
C------------------------------------------------------------
C       Where
C                 P    - The Longitudinal (Axial) Load
C                 M    - The Initial Bending Moment
C                 H    - The Section Depth
C                 B    - The Section Width
C                 C    - The Plastic Centroid or The Geometric
C                        Center of Gravity of The Section
C                 D    - Tensile Reinforcement Depth From Top Fiber
C                 DC   - Compression Reinforcement Depth From Top Fiber
C                 AS   - Area of Tensile Reinforcement
C                 ASC  - Area of Compression Reinforcement
C                 FY   - Yield Stress of Steel
C                 EY   - Yield Strain of Steel
C                 FC   - Concrete Compression Strength (28 Days)
C                 E0   - Concrete Strain for Max Stress
C                 EU   - Ultimate Concrete Strain
C                 FI   - Assumed Initial Strain
C                 E4I  - Assumed Initial Curvature
C                 ALPHA & BETA - Increment Coefficients Depending
C                        on The Desired Accuracy
C------------------------------------------------------------
      WRITE(21.25)
      WRITE(*.25)
25    FORMAT(17X.'MOMENT-CURVATURE'./.19X.'RELATIONSHIP'.//.
     *60('-')./.3X.'CYCLE'.6X.'LOAD'.5X.'MOMENT'.5X.'CURVATURE'.
     *5X.'STRAIN'./.60('-'))
C------------------------------------------------------------
      CONST1-H*FI*2.0
      CONST2-H*FI*0.2
      IF(E4I.GT.CONST1.OR.E4I.LT.CONST2) THEN
        E4I-H*FI
      END IF
      EC-(57000.0)*(SQRT(FC))
      FR-(7.5)*(SQRT(FC))
      ER-(FR)/(EC)
100   CALL TRIG(FI.E4I.H.D.DC.AS.ASC.AI.E1I.E2I.E3I)
      CALL STRESS(EE.FF.E1I.E2I.E3I.FS1I.FS2I.FS3I)
      CALL LOADMOM(AI.B.C.E0.E1I.E2I.E3I.E4I.EU.FC.EC.ER.
     *             FS2I.FS3I.AS.ASC.D.DC.PI.MI)
      DFI-ALPHA*FI+BETA
      DE4I-ALPHA*E4I+BETA
C------------------------------------------------------------
      FIF-FI+DFI
      E4IF-E4I
      CALL TRIG(FIF.E4IF.H.D.DC.AS.ASC.AIF.E1IF.E2IF.E3IF)
      CALL STRESS(EE.FF.E1IF.E2IF.E3IF.FS1IF.FS2IF.FS3IF)
      CALL LOADMOM(AIF.B.C.E0.E1IF.E2IF.E3IF.E4IF.EU.FC.EC.ER.
     *             FS2IF.FS3IF.AS.ASC.D.DC.PIF.MIF)
      FPPARTIAL-(PIF-PI)/DFI
      FMPARTIAL-(MIF-MI)/DFI
C------------------------------------------------------------
      FIE-FI
      E4IE-E4I+DE4I
      CALL TRIG(FIE.E4IE.H.D.DC.AS.ASC.AIE.E1IE.E2IE.E3IE)
      CALL STRESS(EE.FF.E1IE.E2IE.E3IE.FS1IE.FS2IE.FS3IE)
      CALL LOADMOM(AIE.B.C.E0.E1IE.E2IE.E3IE.E4IE.EU.FC.EC.ER.
     *             FS2IE.FS3IE.AS.ASC.D.DC.PIE.MIE)
      EPPARTIAL-(PIE-PI)/DE4I
      EMPARTIAL-(MIE-MI)/DE4I
C------------------------------------------------------------
      E4CHANGE-((M-MI)*(FPPARTIAL)-(P-PI)*(FMPARTIAL))/
     *         ((FPPARTIAL)*(EMPARTIAL)-(EPPARTIAL)*(FMPARTIAL))
      FCHANGE-(M-MI-(EMPARTIAL)*(E4CHANGE))/(FMPARTIAL)
      FNEW-FI+FCHANGE
      E4NEW-E4I+E4CHANGE
      CALL TRIG(FNEW.E4NEW.H.D.DC.AS.ASC.ANEW.E1NEW.E2NEW.E3NEW)
      CALL STRESS(EE.FF.E1NEW.E2NEW.E3NEW.FS1NEW.FS2NEW.FS3NEW)
      CALL LOADMOM(ANEW.B.C.E0.E1NEW.E2NEW.E3NEW.E4NEW.EU.FC.EC.ER.
     *             FS2NEW.FS3NEW.AS.ASC.D.DC.PNEW.MNEW)
      PDIFF-ABS(P-PI)
      MDIFF-ABS(M-MI)
      WRITE(21.200) NCYCLE.PI.MI.FI.E4I
      WRITE(*.200) NCYCLE.PI.MI.FI.E4I
200   FORMAT(2X.I4.6X.F8.2.2X.F8.2.2X.F12.10.2X.F12.10./.60('-'))
      IF(PDIFF.GT.0.001.OR.MDIFF.GT.0.001) THEN
        FI-FNEW
        E4I-E4NEW
      END IF
        NCYCLE-NCYCLE+1
      IF(PDIFF.GT.0.001.OR.MDIFF.GT.0.001) GO TO 100
      WRITE(21.300)
      WRITE(*.300)
300   FORMAT(38X.'By : Naji Al-Mutairi ')
C------------------------------------------------------------
      CLOSE(20)
      CLOSE(21)
      STOP
      END
C***********************************************************
C***********************************************************
C
C    THE SUBROUTINES
C    _______________
C
      SUBROUTINE TRIG (F.E4.H.D.DC.AS.ASC.A.E1.E2.E3)
C      This subroutine calc. the strains in the section
       A-E4/F
       E1-((H-A)/(A))*E4
       IF(AS.GT.0.0) THEN
         E2-((D-A)/(A))*E4
       ELSE
         E2-0.0
       END IF
       IF(ASC.GT.0.0) THEN
         E3-((A-DC)/(A))*E4
       ELSE
         E3-0.0
       END IF
      RETURN
      END
C***********************************************************
      SUBROUTINE STRESS (EE.FF.E1.E2.E3.FS1.FS2.FS3)
C      This subroutine calc. the steel stresses in the section
C      using the strains and the stress-strain diagram
       DIMENSION EE(100).FF(100)
       REAL FS2.FS3
       DO 30 I-1.20
         J-I+1
         IF(E2.GE.EE(I).AND.E2.LT.EE(J)) THEN
             FS2-FF(I)+((FF(I)-FF(J))/(EE(I)-EE(J)))*(E2-EE(I))
         END IF
         IF(E3.GE.EE(I).AND.E3.LT.EE(J)) THEN
             FS3-FF(I)+((FF(I)-FF(J))/(EE(I)-EE(J)))*(E3-EE(I))
         END IF
30     CONTINUE
      RETURN
      END
C***********************************************************
      SUBROUTINE LOADMOM(A.B.C.E0.E1.E2.E3.E4.EU.FC.EC.ER.
     *                   FS2.FS3.AS.ASC.D.DC.P.M)
C      This subroutine calc. the moment and load of the section
      REAL A.B.C.E0.E1.E2.E3.E4.EU.FC.EC.ER.EX
     *       FS2.FS3.AS.ASC.D.DC.P.M
        IF(E1.LT.ER) THEN
          EX-E1
        ELSE
          EX-0.0
        END IF
        IF(E4.LE.E0) THEN
          P-0.85*A*B*FC*((E4/E0)-((1.0/3.0)*((E4/E0)**2.0)))
     *       -FS2*AS+FS3*ASC-(A/E4)*(B)*(EX*(EC*EX))/(2.0)
         M-0.85*(A**2.0)*B*FC*(((2.0/3.0)*(E4/E0))-
     *       ((1.0/4.0)*((E4/E0)**2.0 )))+FS2*AS*(D-A)+
     *       FS3*ASC*(A-DC)-P*(A-C)+
     *       ((A/E4)**2)*(B)*(EC)*((EX)**3)/(3.0)
        END IF
        IF(E4.GT.E0.AND.E4.LE.EU) THEN
          P-0.85*A*B*FC*((2.0/3.0)+(1.0/E4)*((E0-E4)+
     *      ((0.15)/(EU-E0))*((1.0/2.0)*(E0**2.0)+
     *      (1.0/2.0)*(E4**2.0)-(E0*E4))))-FS2*AS+FS3*ASC-
     *      (A/E4)*(B)*(EX*(EC*EX))/(2.0)
         M-0.85*(A**2.0)*B*FC*((5.0/12.0)+(1.0/(E4**2.0))*
     *      ((1.0/2.0)*((E0**2.0)-(E4**2.0))+((0.15)/
     *      (EU-E0))*((E0**3.0)/(3.0)+(E4**3.0)/(3.0)-
     *      (E0)*(E4**2.0)/(2.0))))+FS2*AS*(D-A)+
     *      FS3*ASC*(A-DC)-(P)*(A-C)+
     *      ((A/E4)**2)*(B)*(EC)*((EX)**3)/(3.0)
        END IF
      RETURN
      END
```

Appendix D - Sample Input

```
----------
Input Data
----------

   0.00000   00000.0
   0.00200   60000.0
   0.00600   60400.0
   0.00601    9999
   0.001    60000.0
   8.0    12.0    4.0    7.0    0.0
   0.294    0.0    60000.0    0.002
   4000.0    0.002    0.003
   0.0000000010   0.00000910   0.00001    0.0000000001
```

Appendix E - Sample Output

Output Data

MOMENT-CURVATURE
RELATIONSHIP

CYCLE	LOAD	MOMENT	CURVATURE	STRAIN
0	1.31	1.74	.0000000010	.0000000080
1	-993.32	60513.66	.0000339044	.0001364126
2	-21.15	60123.78	.0000352620	.0001427995
3	-.10	59998.73	.0000353367	.0001429637
4	.00	59999.99	.0000353364	.0001429641

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